Ponzano-Regge model revisited III: Feynman diagrams and Effective field theory

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Abstract

We study the no gravity limit $G_N \to 0$ of the Ponzano-Regge amplitudes with massive particles and show that we recover in this limit Feynman graph amplitudes (with Hadamard propagator) expressed as an abelian spin foam model. We show how the G_N expansion of the Ponzano-Regge amplitudes can be resummed. This leads to the conclusion that the effective dynamics of quantum particles coupled to quantum 3d gravity can be expressed in terms of an effective new non commutative field theory which respects the principles of doubly special relativity. We discuss the construction of Lorentzian spin foam models including Feynman propagators.

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1 Introduction

Spin Foam models offer a rigorous framework implementing a path integral for quantum gravity [1]. They provide a definition of a quantum spacetime in purely algebraic and combinatorial terms and describe it as generalized two-dimensional Feynman diagrams with degrees of freedom propagating along surfaces. Since these models were introduced, the most pressing issue has been to understand their semi-classical limit, in order to check whether we effectively recover general relativity and quantum field theory as low energy regimes and in order to make physical and experimental predictions carrying a quantum gravity signature. A necessary ingredient of such an analysis is the inclusion of matter and particles in a setting which has been primarily constructed for pure gravity. On one hand, matter degrees of freedom allow to write physically relevant diffeomorphism invariant observables, which are needed to fully build and interpret the theory. On the other hand, ultimately, we would like to derive an effective theory describing the propagation of matter within a quantum geometry and extract quantum gravity corrections to scattering amplitudes and cross-sections.

In the present work, we study these issues in the case of a three-dimensional spacetime. Three-dimensional gravity is seen as a toy model in the search of a consistent full quantum general relativity. It is known to be an integrable system carrying a finite number of degrees of freedom. Nevertheless, despite its apparent simplicity, we face many of the same mathematical and interpretational problems than when addressing the quantization of four-dimensional gravity. In this context, the Ponzano-Regge model was actually the first model of quantum gravity ever written [2]. It gives an explicit prescription for the scalar product of Euclidean three-dimensional gravity as a state sum over discretized geometries. As the simplest non-trivial spin foam model, it is the natural arena to investigate the particle couplings to quantum geometry and the semi-classical regime of the theory. Recently, this model has been studied in much detail. The paper [3] has tackled particle insertions and shown how they can be understood as a partial gauge fixing of the state sum model. As a result, explicit quantum amplitudes for massive and spinning particles coupled to gravity were constructed. The work in [4] established an explicit link between the Ponzano-Regge quantum gravity and the traditional Chern-Simons quantization, and identified a κ-deformation of the Poincaré group as the relevant symmetry group of spin foam amplitudes. In this article we follow the same line of thoughts focusing on the relation between spin foam amplitudes and usual Feynman graph evaluations.

The first step is to show that the usual Feynman graph evaluations (with the Hadamard propagator) for quantum field theory on a three-dimensional flat Euclidean spacetime can be identified through a duality transform as spin foam amplitudes of an abelian model. This construction can be extended to the non-abelian case, in which case we describe Feynman diagrams for particles propagating in 3d quantum gravity. The natural mass unit for particles is the Planck defined solely from the Newton constant for gravitation $m_P = 1/\kappa = 1/4\pi G$.

We define a gravity-less limit of the full quantum gravity theory as the $G \to 0$ limit. And we show that the Ponzano-Regge amplitudes constructed in [3] reduce to the amplitudes of the abelian model with the standard Feynman evaluations: as required, the classical limit of matter insertion is the standard quantum field theory.

Then, at $G \neq 0$, the spin foam amplitudes are to be interpreted as providing the Feynman graph evaluation of particles coupled to quantum gravity. We study the perturbative G expansion of the spin foam amplitudes. Remarkably, this expansion can be re-summed and expressed as the Feynman graphs of a non-commutative braided quantum field theory with deformation parameter G, which thus describes the effective theory for matter in quantum gravity. Any deformed Poincaré

theory usually suffers from a huge ambiguity [5] coming from what should be identify as the physical energy and momenta since the introduction of the Planck scale allows non-linear redefinitions. This ambiguity can also be understood as an ambiguity in the identification of the non-commutative space-time. Our work show that the Ponzano-Regge model naturally defines a star product and a duality between space and momenta, therefore no ambiguity remains once we identify quantum gravity as being responsible for the effective deformation of the Poincaré symmetry.

This realizes explicitly, for the first time from first principles, the now popular idea that quantum gravity will eventually lead to an effective non-commutative field theory incorporating the principle of Doubly special relativity [6].

When we also take into account a non vanishing cosmological constant Λ, 3d Euclidean quantum gravity is described by the Turaev-Viro model whose coupling to matter was investigated by Barrett et al. [7, 8]. It is based on the quantum group $\mathfrak{su}_q(2)$. Λ and G now define natural length and mass units, bounding the physical quantities by above and below. We define different classical limits, which correspond to sending some of these units to zero or infinity. Interestingly, they all correspond to the same limit of the deformation parameter $q \to 1$. However they define different limits of the Feynman graph evaluations, depending on which units we use to describe the mass and length of physical objects. We define a hyperbolic spin foam models corresponding to a space-time at $\Lambda \leq 0$. We study the gravity-less limit $G \to 0$ of the Turaev-Viro model and of the hyperbolic state sum model.

Finally, we tackle the issue of causality of the particle amplitudes. Indeed, in the Euclidean framework, we have dealt up to now with the Hadamard propagator. It is non-oriented in time and non-causal. In the Lorentzian spin foam model, it is possible to discuss the different propagators -Wightman, Feynman and Hadamard- and write down the corresponding amplitudes in the non-abelian set-up. This means that we now have the explicit proposal for the amplitudes of the non-commutative quantum field theory describing the effective theory of 3d Lorentzian quantum gravity.

2 Spin Foams as the topological dual of Feynman diagrams: the abelian case

We consider the following Feynman diagram amplitudes associated to an oriented graph Γ for a massive relativistic scalar field in the 3d Euclidean spacetime:

$$I_\Gamma = \int \prod_v d\vec{x}_v \prod_e G_{m_e}(\vec{x}_{t(e)} - \vec{x}_{s(e)}), \tag{1}$$

where v and e label the vertices and the edges of the graph Γ, m_e the mass of the particles living on the edge e and G_m the propagator of the theory. The $\vec{x}$ correspond to the positions of the particles/vertices of the graph (they are 3-vectors). Here we take:

$$G_m(\vec{x}) = \frac{\sin m|x|}{|x|}. \tag{2}$$

Let us insist on the fact that we are using the Hadamard function, and not the Feynman propagator. So we should call *Feynman-Hadamard evaluation*, but we keep it to Feynman evaluation whenever there can not be any confusion. We will later discuss the use and the implementation of the

Feynman propagator in the spinfoam framework, when dealing with the Lorentzian version of the models. In the present section, we show how we can recast the Feynman evaluation as a spinfoam amplitude through a duality transform.

Let us start by switching to the Fourier transform. For this purpose we use the Kirillov formula:

$$G_m(\vec{x}) = m \int_{\mathcal{S}^2} d^2\vec{n}\, e^{im\vec{x}\cdot\vec{n}}, \tag{3}$$

then we change the space of integration from $\mathcal{S}^2$ to $\mathbb{R}^3$ using:

$$\int_{\mathcal{S}^2} d^2\vec{n} = \frac{1}{4\pi m^2} \int_{\mathbb{R}^3} d^3\vec{p}\, \delta(|p| - m). \tag{4}$$

This leads to:

$$I_\Gamma = \int \prod_v d\vec{x}_v \int \prod_e d^3\vec{p}_e \, \frac{\delta(|p_e| - m_e)}{4\pi m_e} \prod_e e^{i\vec{p_e}\cdot(\vec{x}_{t(e)} - \vec{x}_{s(e)})}. \tag{5}$$

Integrating over the $\vec{x}_v$, we finally get:

$$I_\Gamma = \int \prod_e d^3\vec{p}_e \, \frac{\delta(|p_e| - m_e)}{4\pi m_e} \prod_v \delta\left(\sum_{e|v=t(e)} \vec{p}_e - \sum_{e|v=s(e)} \vec{p}_e \right). \tag{6}$$

The $\delta(|p| - m)$ fixes the norm of the momenta to be the mass[1] and the $\delta(\sum \vec{p}_e)$ at the vertices express momentum conservation.

Let us now go to the topological dual and express this amplitude as a Feynman evaluation on the dual graph. A similar duality transformation on Feynman graphs was performed in [9].

More precisely, let us embed the graph Γ in a surface S. This amounts to defining faces f (as sequences of edges) on the graph, each edge belonging to two faces. Then one can define the topological dual, the dual vertices $\bar{v}$ being the faces, the dual edges $\bar{e}$ linking the $\bar{v}$ being associated to the original edges e and the dual faces corresponding to the initial vertices v.

We would like to solve the constraint on the dual faces $\bar{f}$ (i.e at the vertices v) imposed by the $\delta(\sum_{\bar{e}\in\partial\bar{f}} \epsilon_{\bar{f}}(\bar{e})\vec{p}_{\bar{e}})$ (where $\epsilon_{\bar{f}}(\bar{e})$ is a sign recording the orientation of the edge $\bar{e}$ relatively to the face $\bar{f}$). From the algebraic topology point of view, p is a field valued on the edges $\bar{e}$, so it is a 1-form. Its derivate ∂p is a 2-form ie a field valued on faces $\bar{f}$. More precisely:

$$(\partial p)_{\bar{f}} = \langle \partial\bar{f}, p \rangle = \sum_{\bar{e}\in\partial\bar{f}} \epsilon_{\bar{f}}(\bar{e})\vec{p}_{\bar{e}}.$$

Therefore we are imposing that $\partial p = 0$. If the (co)homology group $H^1(S) = H_1(S)$ is trivial, all closed 1-forms are exact. Then there exists a field $\vec{u}$ valued on vertices $\bar{v}$ such that $p = \partial u$ i.e $p_{\bar{e}} = u_{t(\bar{e})} - u_{s(\bar{e})}$, whith $t(\bar{e}), s(\bar{e})$ being the starting and terminal vertices of eb. More generally H^1 is generated by the cycles of S. So if $S \sim \mathcal{S}^2$, everything is straightforward. But if the genus of S is $g \geq 1$, then we also need variables $\vec{u}_C$ attached to the cycles of S. These cycles C are defined, similarly to faces, as sequences of edges $\bar{e}$. And now the solution to ∂X are given by:

$$p_{\bar{e}} = u_{t(\bar{e})} - u_{s(\bar{e})} + \sum_{C\ni\bar{e}} \epsilon_C(\bar{e}) u_C, \tag{7}$$

[1] $\delta(|\vec{p}| - m)/2m$ is equivalent to the usual $\delta(|\vec{p}|^2 - m^2)$.

where $\epsilon_C(\bar{e}) = \pm$ depending on the orientation of the dual edge $\bar{e}$ along the cycle C.

A simple counting argument could reproduce the same output. Indeed we have E (number of edges) variables $p_{\bar{e}}$ and $V-1$ (V number of vertices) constraints. So we would need $E-(V-1) = F-\chi+1$ variables to parametrize the solutions to this linear system, with χ the Euler characteristic of the surface S. As we use variables associated to the faces (i.e the dual vertices), let us allow for a closure relation on these face variables, so we would need $F-\chi+2$ variables. If S is orientable then, this gives $F+2g$ variables: one variable per face and one variable per cycle. The same holds for non-orientable surfaces with $F+g$ variables.

Finally the Feynman graph evaluation reads:

$$I_\Gamma = \int_{\mathbb{R}^3} \prod_{\bar{v}} d\vec{u}_{\bar{v}} \prod_{C\,cycles} d\vec{u}_C \prod_e \frac{\delta(|\vec{p}_{\bar{e}}| - m_e)}{4\pi m_e} \qquad \text{with} \qquad p_{\bar{e}} = u_{t(\bar{e})} - u_{s(\bar{e})} + \sum_{C \ni \bar{e}} \epsilon_C(\bar{e}) u_C. \tag{8}$$

The reader can find examples for $S = \mathcal{S}^2$ and $S = T^2$ in appendix.

More generally, let us consider a triangulated surface S and the graph Γ embedded in the triangulation (but not necessarily covering the whole triangulation). We could more generally consider a generic cellular decomposition of S without loss of generality. We attach variables to every face/triangle, or equivalently to all dual vertices $\bar{v}$, and to every cycle of S. Then the Feynman evaluation simply reads:

$$I_\Gamma = \int_{\mathbb{R}^3} \prod_{\bar{v}\in S} d\vec{u}_{\bar{v}} \prod_{C \text{ cycles of } S} d\vec{u}_C \prod_{e\in\Gamma} \frac{\delta\left(|\vec{u}_{t(\bar{e})} - \vec{u}_{s(\bar{e})} + \sum_{C\ni\bar{e}} \epsilon_C(\bar{e})\vec{u}_C| - m_e\right)}{4\pi m_e} \prod_{e\in S\setminus\Gamma} \delta(\vec{u}_{t(\bar{e})} - \vec{u}_{s(\bar{e})}), \tag{9}$$

where the δ functions on the edges in $S\setminus\Gamma$ allows to 'collapse' the triangulation of S down to the graph Γ.

We now embed the Feynman diagram in the three-dimensional spacetime and recast the Feynman evaluation in terms of the three-dimensional objects. Let us start with a three-dimensional triangulation Δ (or more generally a cellular decomposition), on which the graph Γ is drawn. We now would like to express the Feynman amplitude in terms of variables living on the structures (vertices, edges, faces or 3-cells) of Δ of its dual. We remind that the (2-skeleton of the) topological dual of Δ is called the *spin foam*.

Let us choose a surface $S \subset \Delta$ in which Γ is faithfully embedded [2] and rewrite the formula (9). S is a collection of faces f (triangles more exactly if Δ is strictly speaking a triangulation) to which we associate vectors $\vec{u}_f$. Equivalently, one can consider the dual edges e^* transverse to the faces f and the corresponding variables $\vec{u}_{e^*}$. Then one can directly re-write the Feynman evaluation as:

$$I_\Gamma = \int_{\mathbb{R}^3} \prod_{f\in S} d\vec{u}_f \prod_{C \text{ cycles of } S} d\vec{u}_C \prod_{e\in\Gamma} \frac{\delta\left(|\vec{u}_{t(e)} - \vec{u}_{s(e)} + \sum_{C\ni e} s_C(e)\vec{u}_C| - m_e\right)}{4\pi m_e} \prod_{e\in S\setminus\Gamma} \delta(\vec{u}_{t(e)} - \vec{u}_{s(e)}), \tag{10}$$

where $t(e)$ and $s(e)$ are the two *faces* of S adjacent to the edge $e \in S$, and where $e \in C$ means that the link $\bar{e}$ dual to the edge e on the surface S belongs to the cycle C. We give an example for $S = \mathcal{S}^2, \Delta = \mathcal{S}^3$ in appendix.

Let us now show how such an amplitude arises from spin foam models. Once the three-dimensional triangulation Δ is chosen, one can define the partition function of the (topological)

[2] We require that the components of S when the graph Γ is removed are all homeomorphic to a 2d disk.

spin foam model corresponding to the quantization of an abelian BF theory for the gauge group $\mathbb{R}^3$ on the dual of Δ. One associate group variables -here vectors- $\vec{u}_{e^*}$ to every dual oriented edge e^* (or equivalently to every face of Δ) and one requires that the holonomies around all dual faces f^* are trivial:

$$Z_\Delta^{(ab)} = \int_{\mathbb{R}^3} \prod_{e^*} d\vec{u}_{e^*} \prod_{f^*} \delta(\vec{u}_{f^*}), \tag{11}$$

where we have introduced the notation

$$\vec{u}_{f^*} \equiv \sum_{e^* \in \partial f^*} \epsilon_{f^*}(e^*) \vec{u}_{e^*}.$$

In order to introduce particles, it is more convenient to expand the δ function in the partition function using (3). Then using the fact that a dual face f^* is literally an edge $e \in \Delta$, the spin foam amplitude reads:

$$Z_\Delta = \int \prod_{e^* \in \Delta^*} d\vec{u}_{e^*} \int_{R_+} \prod_{e \in \delta} l_e dl_e \prod_{e \in \delta} \frac{\sin l_e |\vec{u}_e|}{|\vec{u}_e|}. \tag{12}$$

Usually, a spin foam amplitude is presented as a product of weights depending solely on the l_e variables after integration over the holonomies $\vec{u}_{e^*}$. Nevertheless, this first order formalism, keeping both $\vec{u}_{e^*}$ and l_e variables, is necessary to describe the insertion of particles. More precisely, inserting a particle on a given edge e with mass m_e corresponds to the observable:

$$\mathcal{O}_e^{(ab)}(m_e) = \frac{\delta(l_e)}{l_e^2} \times \frac{\delta\left(|\vec{u}_e| - m_e\right)}{4\pi m_e}. \tag{13}$$

One can obtain this expression as the abelian limit of the Ponzano-Regge spinfoam model discussed in the next section whose particle insertions have been described in [3]. Then inserting a whole Feynman diagram corresponds to putting particles on all edges of a given graph Γ. Inserting the corresponding observable into the partition function gives:

$$Z_\Delta^{(ab)}(\Gamma, \{m_e\}) = \int \prod_{e^*} d\vec{u}_{e^*} \prod_{e \in \Gamma} \frac{\delta\left(|\vec{u}_e| - m_e\right)}{4\pi m_e} \prod_{e \notin \Gamma} \delta(\vec{u}_e). \tag{14}$$

Now one would like to show that the previous spin foam amplitude (14) reproduces the Feynman evaluation (10). The two expressions already look very similar. But we would like to be more exact and identify a particular 3d triangulation Δ (or more precisely a class of triangulations) such that the equality $Z_\Delta(\Gamma, \{m_e\}) = I_\Gamma(m_e)$ holds.

Let us start by a couple of remarks on the spin foam amplitude. First, the BF partition function is topologically invariant, meaning that two triangulations with the same topology will yield the same spin foam amplitude. Then, following the analysis of [10, 3], we know that the spin foam amplitude is in general ill-defined and that it requires a gauge-fixing in order to give a finite amplitude. Roughly, the origin of the divergence is that the δ function appearing in the spin foam partition function are redundant and, using the topological invariance, we can generally choose a smaller set of conditions to impose that the curvature is flat. The completely gauge-fixed amplitude for an arbitrary triangulation Δ corresponds to the smallest triangulation of the same topology and compatible with the boundary data (and the graph Γ being considered as boundary).

The main result is that the equality is achieved for any 3d triangulation Δ topologically equivalent to the 3-sphere:

$$Z_\Delta(\Gamma, \{m_e\}) = I_\Gamma(m_e). \tag{15}$$

A general proof in the non-abelian case will be presented in the next section. Let us still explain the reason why it works. The reader can also find explicit examples in appendix.

We start with a graph Γ embedded in Δ, and we introduce a framing i.e a surface $S \subset \Delta$ in which Γ is faithfully embedded. First we consider the case with no non-contractible cycles, $S \sim \mathcal{S}^2$. The triangulation of the surface S directly provides us with a triangulation of $\mathcal{S}_3$: $\mathcal{S}_3$ can be realized by taking two copies of the 3-ball glued back together along $\mathcal{S}_2$. Then simply choosing as triangulation Δ the initial triangulation for S, the two expressions (10) and (14) simply match.

The case with cycles is a bit more subtle. Once again, we start with the graph Γ drawn on the surface S, on which we have chosen certain sequences of dual links $\bar{e}$ to represent the non-contractible cycles. Looking at the Feynman evaluation, we associate a variable $\vec{u}_C$ to each cycle. To match the spin foam amplitude (14), we associate a face of the 3d triangulated manifold Δ to each cycle: the boundary of this face being the given cycle. More precisely, the cycles of an orientable surface S can be split into a set of cycles a_i and their dual cycle b_i. We construct the 3d spacetime by taking two copies of S, one for which we add faces corresponding to the cycles a_i and the other for which we turn the cycles b_i into faces. The added faces might not be triangles at first, but we can triangulate them in the obvious way. Gluing these two filled-up copies of S results into a 3d triangulation once again topologically equivalent to the 3-sphere $\mathcal{S}_3$. The idea is that the interior of S with added faces along one set of cycles, a or b, is topologically a 3-ball. The point is that the triangulation resulting from gluing the two copies of S with added faces makes the equality between (10) and (14) straightforward.

Here we have found one particular triangulation Δ_0 for each Feynman diagram Γ such that $\widetilde{Z}_\Gamma(\Delta_0) = I_\Gamma$. Nevertheless, due to the topological properties of the spin foam models for BF theories, it follows that the same statement is true for any triangulation Δ topologically equivalent to Δ_0 i.e which can be constructed out of Δ_0 by a sequence of Pachner moves (without modifying the graph Γ).

Let us conclude that this equality between the spin foam amplitude and the Feynman evaluation allows us to identify the momentum $\vec{p}_e$ of the particle living on an edge e of the triangulation with the holonomy $\vec{u}_e = \vec{u}_{f^*}$ around the dual face to that edge: this shows us how to encode a particle as geometrical data.

Let us also point out that the Feynman evaluation corresponds to the simplest topology - the one of the 3-sphere. The spin foam framework allows to generalize these Feynman graph amplitudes to arbitrary topologies. It should be interesting to understand better what effect the non trivial topology of the ambient manifold has on Feynman graph evaluation.

In the next section, we will generalize our framework to the non-abelian context of the Ponzano-Regge spinfoam model for 3d quantum gravity, and analyze the Feynman evaluation which results from inserting particles in the quantum gravity theory. This will lead us in the following section to show that the no-gravity limit of the Ponzano-Regge reproduces the abelian spinfoam model and the usual classical Feynman graph evaluation and to identify the Ponzano-Regge amplitudes as providing a perturbative expansion in the gravity coupling which is interpreted as QFT amplitudes on a non-commutative geometry.

3 Particle Insertions in Ponzano-Regge Spinfoam Gravity

In [3] the general form of Feynman graph amplitude for spinning particles in the Ponzano-Regge model has been written.

In this section we focus on the case of spinless particle (a discussion of the spinning case is included in the appendix), we first recall briefly the general construction and then compute explicitly the Feynman diagram amplitudes of particles coupled to three dimensional Euclidean gravity in a form allowing us to take the "no gravity" limit i.e where the Newton constant $G_N \to 0$. The main result of this section (see eq. (39, 40)is a explicit computation of the Ponzano-Regge amplitude which allow a comparison with the usual Feynman graph amplitude computed in the previous section. A similar result has been very recently obtained independently, in the context of the Turaev-Viro model, by Barrett et al. [8].

We start from a triangulation Δ of our spacetime M and consider also the dual Δ^*: dual vertices, edges and faces correspond respectively to tetrahedra, faces and edges of Δ. We choose our Feynman graph to be embedded in the triangulation Δ such that edges of Γ are edges of the triangulation. Each edge of Γ is labelled by an angle $\theta \in [0, \pi]$

$$\theta = \kappa m, \qquad \kappa = 4\pi G_N \tag{16}$$

where G_N is the Newton constant, κ is the inverse Planck mass and m is the mass of the particle.

We choose a group G, here SU(2), and assign group elements g_{e^*} to all dual edges e^* of the triangulation. We constrain the holonomies around dual faces $f^* \sim e$ to be flat if there is no particle and we constraint it to be in the conjugacy class θ_e if e is an edge of Γ. More precisely, let us note $G_e = G_{f^*}$ the product of the group elements around a dual face (or plaquette) $f^* \sim e$:

$$G_e = G_{f^*} = \prod_{e^* \in \partial f^*} g_{e^*}^{\epsilon_{f^*}(e^*)},$$

where $\epsilon_{f^*}(e^*)$ records the orientation of the edge e^* in the boundary of the (dual) face f^*. The amplitude is well defined once we chose a gauge fixing. In order to do so we choose T a maximal tree of $\Delta \setminus \Gamma$ and T^* a maximal tree of Δ^* [10]. Then the partition function reads:

$$\mathcal{Z}_M(\Gamma_\theta) = \int \prod_{e^* \notin T^*} dg_{e^*} \prod_{e \notin T \cup \Gamma} \delta(G_e) \prod_{e \in \Gamma} \Delta(\theta_e) \delta_{\theta_e}(G_e), \tag{17}$$

where dg is the normalized Haar measure and $\delta(g)$ the corresponding delta function on G and $\Delta(\theta) \equiv \sin(\theta)$. The partition function contains a factor $\prod_e \Delta(\theta_e)$ which can be factor out, this factor is important when we consider the no gravity limit. In this section in order to simplify notation we will work with the reduced partition function

$$\widetilde{\mathcal{Z}}_M(\Gamma, \theta) \equiv \frac{\mathcal{Z}_M(\Gamma, \theta)}{\prod_e \Delta(\theta_e)}.$$

Also, $\delta_\theta(g)$ fixes the group element g to be in the conjugacy class[3]labelled by θ. This fixes a non-zero deficit angle around the edge e, which corresponds to the geometrical picture of a particle in a 3d spacetime of mass $4\pi G m = \theta$ (e being the trajectory of the particle). More precisely, the

[3]The Cartan subgroup H of SU(2) is the group of diagonal matrices

$$h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

distribution $\delta_\theta(g)$ is defined by:

$$\forall f, \int_G dg\, \delta_\theta(g) f(g) = \int_{G/H} dx\, f(x h_\theta x^{-1}), \tag{18}$$

where dg and dx are normalized invariant measures. The amplitude (17) does not depend on the choice of the triangulation and the gauge fixing trees, but only on the topology of the manifold M and the embedding of Γ in it [4].

We can expand the δ functions in terms of characters

$$\begin{aligned} \delta(g) &= \sum_j d_j \chi_j(g), \\ \delta_\theta(g) &= \sum_j \chi_j(h_\theta)\chi_j(g), \text{ with } \chi_j(\theta) \equiv \chi_j(h_\theta) = \frac{\sin d_j\theta}{\sin\theta}, \end{aligned} \tag{19}$$

$d_j = 2j+1$ being the dimension of the spin j representation. We eventually perform the integration over g_{e^*} in order to obtain a state sum model

$$\widetilde{\mathcal{Z}}_M(\Gamma_\theta) = \sum_{\{j_e\}} \prod_{e\notin\Gamma} d_{j_e} \prod_{e\in\Gamma} \chi_{j_e}(h_{\theta_e}) \prod_{e\in T} \delta_{j_e,0} \prod_t \left\{ \begin{array}{ccc} j_{e_{t_1}} & j_{e_{t_2}} & j_{e_{t_3}} \\ j_{e_{t_4}} & j_{e_{t_5}} & j_{e_{t_6}} \end{array} \right\}, \tag{20}$$

where the summation is over all edges of Δ and the product of normalized 6j symbols is over all tetrahedra t. For each tetrahedron, the admissible triples of edges, e.g. $(j_{e_{t_1}}, j_{e_{t_2}}, j_{e_{t_3}})$, correspond to faces of this tetrahedra. The factor $\delta_{j,0}$ comes from the gauge fixing, it eliminates the sum over j_e, $e \in T$.

Let us underline the fact that the spin foam amplitudes, like (20), are purely algebraic constructions built with dimensionless quantities. The gravitational coupling constant G_N does not directly enter into the partition function. It appears as a unit used to turn the dimensionless quantities (like the angle θ) into physical properties of the matter and particles (like the mass m).

Starting from the definition (20) we can explicitly compute these amplitudes. First, in order to make contact with usual Feynman graph amplitudes we have to restrict the topology of the spacetime to be trivial, so $M = \mathcal{S}^3$. Also, to warm up, we first consider the case where Γ is a spherical graph which can be embedded in $\mathcal{S}^2$. Lets start with the example of Γ being a tetrahedral graph embedded in $\mathcal{S}^2 \subset \mathcal{S}^3$. We denote by $I = 1,..,4$ the vertices of this graph and $e = (IJ)$ the edges of this graph. Since the amplitude does not depend on the triangulation we are free to chose it. The simplest triangulation of $\mathcal{S}^3$ we can chose in which the graph can be embedded consist of two tetrahedra. One of the tetrahedra gives a triangulation of the interior of $\mathcal{S}^2$ the one other gives a triangulation of the exterior of $\mathcal{S}^2$ the three sphere is obtained by gluing the two balls together. With this graph and triangulation, no gauge fixing is needed, and the corresponding amplitude reads

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \sum_{\{j_{IJ}\}} \prod_{I<J} \chi_{j_{IJ}}(\theta_{IJ}) \left\{ \begin{array}{ccc} j_{34} & j_{24} & j_{23} \\ j_{12} & j_{13} & j_{14} \end{array} \right\}^2. \tag{21}$$

which correspond to rotations of angle 2θ around a given axis (usually the z axis). Every group element is conjugate to such an element. The residual action on $H = \mathrm{U}(1)$ is given by the Weyl group. It is $\mathbb{Z}_2$, since h_θ and $h_{-\theta}$ are conjugated.

The $6j$ square can be written as a group integral

$$\left\{ \begin{array}{ccc} j_{34} & j_{24} & j_{23} \\ j_{12} & j_{13} & j_{14} \end{array} \right\}^2 = \int \prod_I dg_I \prod_{I<J} \chi_{\bar{\jmath}_{IJ}}(g_I g_J^{-1}), \tag{22}$$

where we have introduce the notation $\bar{\jmath}_{IJ} \equiv j_{KL}$, with I, J, K, L all distinct. $\bar{\jmath}_{IJ}$ label the edges of the graph dual to Γ in $\mathcal{S}^2$. We also denote $\bar{\theta}_{IJ} = \theta_{KL}$. Using the previous evaluation and character expansion (19) we can perform the summation over spins and obtain

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_I dg_I \prod_{I<J} \delta_{\bar{\theta}_{IJ}}(g_J g_I^{-1}). \tag{23}$$

This can be explicitly evaluated [11]

$$\mathcal{Z}_{\mathcal{S}^3}(\Gamma_\theta) = \widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) \prod_{I<J} \sin(\theta_{IJ}) = \frac{\pi^2}{2^5} \frac{1}{\sqrt{\det(\cos\theta_{IJ})}}. \tag{24}$$

Lets now consider a general spherical graph $\Gamma \subset \mathcal{S}^2$. In general Γ is not a triangulation of $\mathcal{S}^2$ but we can add edges to it in order to have a regular triangulation of $\mathcal{S}^2$, we denote it Γ_Δ. Once this is done we write as before $\mathcal{S}^3$ as the gluing of two 3-balls, one being the interior of $\mathcal{S}^2$ the other the exterior. We can extend the triangulation Γ_Δ of $\mathcal{S}^2$ to a triangulation of the interior ball by adding interior edges and interior vertices and gluing corresponding tetrahedra[4]. We can choose the triangulation of the exterior ball to be the same as the interior ball with reversed orientation.

Starting from the amplitude (20), we can perform the summation over all edges of the triangulation that do not belong to $\mathcal{S}^2$, we obtain

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \sum_{\{j_e\}} \prod_{e\notin\Gamma} d_{j_e} \prod_{e\in\Gamma} \chi_{j_e}(h_{\theta_e}) |\langle B^3|\Gamma_\Delta, j_e\rangle|^2 \tag{25}$$

where the summation is only over boundary edges and $\langle B^3|\Gamma, j_e\rangle$ is the Ponzano-Regge amplitude associated with the 3-ball, with (Γ, j_e) on its boundary[5]. This amplitude can be understood as the physical scalar product between a spin network state and a 'Hartle-Hawking' state associated with a manifold with boundary. This amplitude can be easily computed and is given by the evaluation of the corresponding spin network. More precisely, let's denote $\overline{\Gamma}$ the graph dual to Γ in $\mathcal{S}^2$ and $\overline{\Gamma}_\Delta$ the graph dual in $\mathcal{S}^2$ to Γ_Δ. The edges of $\overline{\Gamma}_\Delta$ are denoted $\bar{e}$ and are in one to one correspondence

[4]For instance, let's choose an order of the vertices $V = 0, 1, \cdots, n$ such that vertices I and $I+1$ belong to neighboring triangles (sharing an edge). We choose 0 as our reference vertex, then we consider the vertex 1. If 01 is already an edge of our boundary triangulation we do nothing and consider the vertex 2. If 01 is not an edge of our triangulation we add it in the interior of the ball and consider the tetrahedron which consists of this new edge and of the two neighbor triangles containing 0 and 1. And then we continue the procedure. This will provide a triangulation of the 3-ball which extends the triangulation of S^2 and possesses no interior vertex.

[5]We can extend the definition of Ponzano-Regge amplitude to the case of manifold with boundary $\partial M = \Sigma$ with a colored graph drawn on the boundary triangulation to be

$$\langle M|\Gamma, j_e\rangle = \sum_{\{j_e\}} \prod_e d_{j_e} \prod_{e\in T} \delta_{j_e,0} \prod_t \left\{ \begin{array}{ccc} j_{e_{t_1}} & j_{e_{t_2}} & j_{e_{t_3}} \\ j_{e_{t_4}} & j_{e_{t_5}} & j_{e_{t_6}} \end{array} \right\}. \tag{26}$$

where the summation is only over internal edges.

with edges of Γ_Δ. Given a coloring $j_{\bar{e}} = j_e$ of the edges of $\overline{\Gamma}_\Delta$ we can consider the spin network functional

$$\Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(g_{\bar{e}}) \tag{27}$$

which is a gauge invariant function on $G^{|E|}$ with the gauge group acting at vertices. Then

$$\langle B^3 | \Gamma_\Delta, j_e \rangle = \Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(1). \tag{28}$$

The modulus square of this amplitude can be expressed as an integral

$$|\Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(1)|^2 = \int \prod_{\bar{v}} dg_{\bar{v}} \prod_{\bar{e}} \chi_{j_{\bar{e}}}(g_{t(\bar{e})} g^{-1}_{s(\bar{e})}), \tag{29}$$

where $s(\bar{e}), t(\bar{e})$ denote the starting and terminal vertices of the oriented edge $\bar{e}$. We can now perform the summation over $j_{\bar{e}}$ in (25) in order to get

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{\bar{v}} dg_{\bar{v}} \prod_{\bar{e} \in \overline{\Gamma}} \delta_{\theta_{\bar{e}}}(g_{t(\bar{e})} g^{-1}_{s(\bar{e})}) \prod_{\bar{e} \notin \overline{\Gamma}} \delta(g_{t(\bar{e})} g^{-1}_{s(\bar{e})}). \tag{30}$$

We can easily integrate out all the delta functions associated with edges of $\overline{\Gamma}_\Delta$ not in $\overline{\Gamma}$ in order to finally obtain

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{\bar{v} \in \Gamma} dg_{\bar{v}} \prod_{\bar{e} \in \Gamma} \delta_{\theta_{\bar{e}}}(g_{t(\bar{e})} g^{-1}_{s(\bar{e})}), \tag{31}$$

which is the desired result.

We now consider the general case of a graph which is not necessarily spherical but which can be embedded in a Riemann surface of genus g, $\Gamma \subset \Sigma_g$. As before we can add edges to Γ in order to have a regular triangulation of Σ_g, denoted Γ_Δ. Once this is done we write $\mathcal{S}^3$ as the gluing of two handlebodies of genus g: $\mathcal{S}^3 = H_g \sharp_{\Sigma_g} H_g^*$. The meridians of the interior handlebody draw a set of $a_i (i = 1, \cdots, g)$ cycles on Σ_g, the meridians of the exterior handlebody draw a set of $b_i (i = 1, \cdots, g)$ cycles on Σ_g. The a cycles intersect transversally the b cycles and together they form a base of $H^1(\Sigma_g)$. We can extend the triangulation Γ_Δ of Σ_g to a triangulation of the interior handlebody H_g. In order to do so we first have to choose a representative of each meridian as a cycle of edges in Γ_Δ. Each edge $e \in \Gamma_\Delta$ belongs to one of the cycle a_i or to none. We introduce an index $\imath(e)$ to keep track of this, where $\imath(e) \in 0, 1, \cdots, g$ is equal to 0 if it doesn't belong to any cycle and it is equal to i if it belongs to the cycle a_i. Each meridian a_i is the boundary of a meridian disk D_i cutting the handlebody. We choose a triangulation Δ_i of D_i which matches the triangulation of a_i on its boundary. Once this is done we cut H_g along the disks D_i and consider $H_g - \cup_i D_i$. This is a three ball, its boundary contains 2 copies of each meridian disk D_i which have to be identified to reconstruct H_g. Moreover, the triangulation Γ_Δ and the triangulation of the meridian disk induces a triangulation $\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*$ of the boundary $\mathcal{S}^2$ of this ball. We can, as previously, extend this 2-dimensional triangulation to a three dimensional triangulation of the ball. Gluing back the ball along D_i we obtain a triangulation of H_g. For the exterior handlebody H_g^* we do the same except that we have to exchange the cycles a_i with the cycles b_i. In this case we denote $\tilde{\imath}(e)$ the index specifying which b cycle e belongs to.

Starting from the amplitude (20) we can perform the summation over all edges of the triangulation that do not belong to Σ_g, we obtain

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \sum_{\{j_e\}} \prod_{e \notin \Gamma} d_{j_e} \prod_{e \in \Gamma} \chi_{j_e}(h_{\theta_e}) \langle \Delta_\Gamma, j_e | H_g^* \rangle \langle H_g | \Delta_\Gamma, j_e \rangle \tag{32}$$

where the summation is only over boundary edges and $\langle H_g|\Gamma, j_e\rangle$ is the physical scalar product between the spin network state $|\Gamma, j\rangle$ and the 'Hartle-Hawking' state $|H_g\rangle$. This scalar product can be computed, it is given by

$$\langle H_g|\Gamma_\Delta, j_e\rangle = \int \prod_{i=1}^{g} da_i \, \Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(a_{\imath(\bar{e})}^{\epsilon(\bar{e})}) \tag{33}$$

where, $\Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(g_{\bar{e}})$ is the spin network functional, $\imath(\bar{e})$ label which cycle e belongs to, it is 0 if it belongs to none, and $a_0 \equiv 1$. $\epsilon(\bar{e}) = +1$ if e as the same orientation than a_i and $\epsilon(\bar{e}) = -1$ otherwise. The proof of this evaluation goes as follows: the idea is to express, in terms of amplitudes, the fact that H_g is the gluing of a ball along the meridians disks. The edges of the triangulation Γ_Δ of H_g are colored by spins j_e, the triangulation Δ_i of D_i carries additional edges $\tilde{e}_i$ colored by $j_{\tilde{e}_i}$. The triangulation of the ball $\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*$ is then colored by spins $j_e, j_{\tilde{e}_i}$ with the additional condition that the spin coloring Δ_i and Δ_i^* are the same. The handlebody amplitude is then obtained by summing the ball amplitude over all spins $j_{\tilde{e}_i}$

$$\langle H_g|\Gamma_\Delta, j_e\rangle = \sum_{j_{\tilde{e}_i}} d_{j_{\tilde{e}_i}} \langle B^3|\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*, j_e, j_{\tilde{e}_i}\rangle \tag{34}$$

We know that $\langle \mathcal{S}^3|\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*, j_e, j_{\tilde{e}_i}\rangle$ is just given by the evaluation of a spin network based on a graph dual to $\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*$. Moreover, this spin network is such that each spin $j_{\tilde{e}_i}$ appears twice. We can therefore express the evaluation as an integral over group elements associated to the disk's vertex $\tilde{v}$ which are in the interior of the D_i's.

$$\langle \mathcal{S}^3|\Gamma_\Delta \cup \Delta_i \cup \Delta_i^*, j_e, j_{\tilde{e}_i}\rangle = \int \prod_{\tilde{v}} dg_{\tilde{v}} \Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(g_{\tilde{v}(e)}) \prod_{\tilde{e}_i} \chi_{j_{\tilde{e}_i}}(g_{t_{\tilde{e}_i}} g_{s_{\tilde{e}_i}}^{-1}), \tag{35}$$

where $\tilde{v}(e)$ is the disk's vertex which belong to e. If e doesn't intersect any meridian disk it is understood that $g_{\tilde{v}(e)} = 1$. Now the summation over the spins in (34) produces a δ function for every disk edge. One δ function per disk is eliminated by the gauge fixing. We can then integrate out all the group element $g_{\tilde{v}}$ except one per disk which we call a_i. This gives us the expected formula (33). If we insert this evaluation into (32) we get

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{i=1}^{g} da_i db_i \sum_{\{j_e\}} \prod_{e \notin \Gamma} d_{j_e} \prod_{e \in \Gamma} \chi_{j_e}(h_{\theta_e}) \Phi_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(a_{\imath(\bar{e})}^{\epsilon(e)}) \overline{\Phi}_{(\overline{\Gamma}_\Delta, j_{\bar{e}})}(b_{\tilde{\imath}(\bar{e})}^{\epsilon(e)}). \tag{36}$$

This expression is somehow reminiscent of a string theory amplitude[6]. We can now express the product $\Phi\overline{\Phi}$ as an integral over a product of characters for each edge of Γ_Δ and then perform the

[6] In the case of string theory the genus g amplitude is written as

$$Z = \int_{\mathcal{M}_g} dm d\bar{m} \sum_I \Psi_I(m) \overline{\Psi}^I(\bar{m}), \tag{37}$$

where the integral is over the moduli space of Riemann surface and m is the holomorphic moduli and I labels the space of Holomorphic Virasoro conformal blocks. The expression (36) have the same general form if we exchange the moduli $m, \bar{m}$ by a_i, b_i, the label I by spin label and the Conformal block Ψ by the spin network functional Φ. It may be only an accidental analogy.

summation over the j_e in order to obtain

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{f\subset\Sigma_g} dg_f \int \prod_{i=1}^{g} da_i db_i \prod_{e\in\Gamma} \delta_{\theta_e}(g_{t_e} a_{\imath(e)}^{\epsilon(e)} g_{s_e}^{-1} b_{\tilde{\imath}(e)}^{\epsilon(e)}) \prod_{e\notin\Gamma} \delta(g_{t_e} a_{\imath(e)}^{\epsilon(e)} g_{s_e}^{-1} b_{\tilde{\imath}(e)}^{\epsilon(e)}), \tag{38}$$

where we have switched all the notations back to the triangulation so that t_e and s_e are the two faces adjacent to the edge e (they equivalently are the dual vertices $\bar{v}$ ending the dual edge $\bar{e}$). If one chooses the cycles a, b to lie entirely in Γ, we can integrate out the delta functions associated with edges not in Γ and obtain the result we are looking for:

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{f\subset\Sigma_g} dg_f \int \prod_{i=1}^{g} da_i db_i \prod_{e\in\Gamma} \delta_{\theta_e}(g_{t_e} a_{\imath(e)}^{\epsilon(e)} g_{s_e}^{-1} b_{\tilde{\imath}(e)}^{\epsilon(e)}). \tag{39}$$

This is the generalization of the Feynman evaluation (10) to the non-abelian case when inserting particles in 3d quantum gravity. Further assuming that each edge e belongs to a unique cycle of Σ_g, one can simplify the formula :

$$\widetilde{\mathcal{Z}}_{\mathcal{S}^3}(\Gamma_\theta) = \int \prod_{f\subset\Sigma_g} dg_f \int \prod_{C \text{ cycles of } S} dU_C \prod_{e\in\Gamma} \delta_{\theta_e}\left(g_{t_e} U_{C(e)}^{\epsilon_C(e)} g_{s_e}^{-1}\right). \tag{40}$$

4 The no-gravity limit and a perturbative expansion in G

4.1 The Quantum Field Theory limit of Spin Foams at $G \to 0$

Now we would like to study the "no-gravity" limit of the particle insertion amplitudes of the Ponzano-Regge model. That is the limit when we take the Newton constant to zero i.e. $G_N \to 0$. In three spacetime dimensions, the Planck length reads $l_P \sim \hbar G_N$ while the Planck mass is $m_P \sim 1/G_N$. The usual classical limit is taking $\hbar \to 0$ while keeping $G_N \neq 0$. As $m_P \neq 0$ in this limit, we get an effective deformed Poincaré algebra [12] and recover the framework of Doubly Special Relativity. Here, we consider the alternative limit $G_N \to 0$ while $\hbar$ is kept fixed, in this limit, we expect to recover from a quantum gravity theory the usual quantum field theory framework: we call it the *QFT limit*. More exactly, we want to recover the usual Feynman diagram evaluations of quantum field theory on a flat background as described in the section 2.

More precisely, the limit is defined as $\kappa \to 0$ where κ is the inverse Planck mass defined in (16) as the ratio between the deficit angle and the corresponding mass of a particle. To understand how the different quantities gets renormalized in the limit we parametrize $g \in \mathrm{SU}(2)$ in terms of a Lie algebra element $\vec{u} \in \mathbb{R}^3$. In the fundamental representation we have:

$$g \equiv e^{i\kappa u^i \sigma_i} = \cos(m\kappa) + i\sin(m\kappa)\vec{n}.\vec{\sigma}, \tag{41}$$

where $m = |\vec{u}|$, $\vec{n}$ is the direction of the rotation and $\vec{\sigma}$ the Pauli matrices. in the limit, the angle $\theta = m\kappa$ goes to 0 so that we are in fact considering perturbations around the identity in SU(2) or equivalently the limit in which the group SU(2) goes flat and becomes the abelian group defined by its Lie algebra $\mathfrak{su}(2)$. So it is natural to also call the QFT limit the *abelian limit* for spin foams.

We can expand group elements and their products perturbatively in the parameter κ:

$$g_i \sim \left(1 - \kappa^2 \frac{|\vec{u}_i|^2}{2}\right) + i\kappa\vec{u}_i \cdot \vec{\sigma} + \ldots,$$

$$g_1 g_2 \sim 1 + i\kappa\,(\vec{u}_1+\vec{u}_2)\cdot\vec{\sigma} + \frac{\kappa^2}{2}\left(\vec{u}_1^2+\vec{u}_2^2+2\vec{u}_1\cdot\vec{u}_2+2i(\vec{u}_1\wedge\vec{u}_2)\cdot\vec{\sigma}\right)+\ldots, \tag{42}$$

so that the group multiplication is linear at the first order in κ. In the $\kappa\to 0$ limit, the discrete representations label j becomes a continuous length parameter l parameterizing the representations of $\mathfrak{su}(2)\sim\mathbb{R}^3$ as a group and m is the renormalized mass. The QFT limit is defined as by:

$$d_j=\frac{l}{\kappa}, \qquad \theta = m\kappa, \qquad \kappa\to 0, \qquad l, m \text{ fixed} \tag{43}$$

In this limit the integral over the group becomes integral over the Lie algebra

$$\int_G dg \sim_{\kappa\to 0} \kappa^3 \int_{\mathbb{R}^3}\frac{d^3\vec{u}}{2\pi^2} \tag{44}$$

where the normalization factor insures that

$$\int \frac{d^3\vec{u}}{2\pi^2} f(\vec{u}) = \frac{2}{\pi}\int_0^\infty dm\, m^2 \int_{\mathcal{S}^2} d^2\vec{n} f(m\vec{n}), \tag{45}$$

with $d\vec{n}$ the normalized measure on the sphere[7]. Summation over j becomes an integral

$$\sum_j \sim_{\kappa\to 0} \frac{1}{\kappa}\int_0^\infty dl, \tag{47}$$

and we recover the usual classical Hadamard propagator (2) as the abelian limit of the SU(2) character.

$$\chi_j(g) = \frac{\sin d_j\theta}{\sin\theta} \sim_{\kappa\to 0} \frac{1}{\kappa}\frac{\sin l|\vec{u}|}{|\vec{u}|}. \tag{48}$$

The Ponzano-Regge partition function (17) is given by

$$\mathcal{Z}_M(\Gamma_\theta) = \int \prod_{e^*\notin T^*} dg_{e^*} \prod_{e\notin T\cup\Gamma}\delta(G_e)\prod_{e\in\Gamma}\delta_{\theta_e}(G_e)\Delta(\theta_e). \tag{49}$$

Let's first consider the case of a closed manifold without particles and expand the delta functions in terms of characters, this gives

$$\mathcal{Z}_M = \sum_{j_e}\prod_e d_{j_e}\int\prod_{e^*} dg_{e^*}\prod_e \chi_{j_e}(G_e). \tag{50}$$

Now taking the limit $\kappa\to 0$ is straightforward. One uses the fact that the product $g_e = g_{f^*} = \prod_{e^*} g_{e^*}$ is an abelian product at the first order in κ then takes the limit of the characters. This leads to:

$$Z = \lim_{\kappa\to 0}\mathcal{Z} = \int\prod_{e^*\notin T^*}\frac{d^3\vec{u}_{e^*}}{2\pi^2}\int_{R_+} l_e dl_e \prod_{e\notin T}\frac{\sin l_e|\vec{u}_e|}{|\vec{u}_e|} = \int_{\mathbb{R}^3}\prod_{e^*} d\vec{u}_{e^*}\prod_e \delta(\vec{u}_e), \tag{51}$$

[7]The normalization comes from the Weyl integration formula

$$\int dg f(g) = \frac{2}{\pi}\int_0^\pi d\theta \sin^2(\theta)\int_{G/H} dx f(xgx^{-1}) \tag{46}$$

with dg, dx being normalized invariant measures

where the vectors $\vec{u}_{e^*}$ are the Lie algebra counterpart of the group elements g_{e^*}:

$$\vec{u}_{e^*} \equiv m_{e^*}\vec{n}_{e^*}, \qquad \vec{u}_e \equiv \vec{u}_{f^*} \equiv \sum_{e^*\in\partial f^*} \vec{u}_{e^*}.$$

We see that we naturally recover the abelian model studied previously. In the limit we have to keep track of the factor κ. For each edge e not in T, there is a factor $1/\kappa^3$ (one coming from $\sum_j$, one from d_j and one from χ_j) and for each dual edge e^* not in T^* there is a factor κ^3. Overall this means that κ comes with the power $3\chi(M)$, where $\chi(M) = \sigma_0 - \sigma_1 + \sigma_2 - \sigma_3$ is the Euler characteristic of Δ and σ_i are the number of i-cells of the triangulation. For a closed three manifold $\chi(M) = 0$ so no factor κ appears in the limit of the partition function.

A particle insertion of mass θ_e on an edge e, or dual face f^*, corresponds to inserting the following observable [3] in the partition function (50):

$$\mathcal{O}_e(\theta_e) = \delta_{j_e 0} \times \delta_{\theta_e}(g_e) \times \Delta(\theta) \tag{52}$$

where $\Delta(\theta) = \sin\theta$ and the distribution $\delta_\theta(g)$ is defined in (18). Inserting this operator in $\mathcal{Z}$ relaxes the constraint $\delta(g)$ by replacing it by the weaker constraint $\delta_\theta(g)$, which gives back the formula (17).

For a particle insertion on the edge e with mass m_e, we first observe that the limit of $\delta_\theta(g)$ is $\delta_m(\vec{u})$, which is a distribution satisfying the condition (obtained as the limit of (18)):

$$\int_{\mathbb{R}^3} \frac{d^3\vec{u}}{2\pi^2}\, \delta_m(|\vec{u}|) f(\vec{u}) = \frac{1}{\kappa^3} \int_{\mathcal{S}^2} d^2\vec{n}\, f(m\vec{n}) \tag{53}$$

so $\delta_m(\vec{u}) = \frac{\pi}{2\kappa^3 m^2}\delta(|\vec{u}| - m)$. Note that we have the analog of identity (19)

$$\int_0^\infty dl \frac{\sin lm}{m} \frac{\sin l|\vec{u}|}{|\vec{u}|} = \frac{\pi}{2m^2}\delta(|\vec{u}| - m). \tag{54}$$

The term δ_{j0} in (52) kills the summation $\sum_j d_j \chi_j(G)$ which becomes $1/\kappa^3 \int dl\, l \frac{\sin l|\vec{u}|}{|\vec{u}|}$ in the abelian limit. So the abelian limit of δ_{j0} is $\kappa^3 \frac{\delta(l)}{l^2}$. Eventually, the abelian limit of $\Delta(\theta)$ is κm. This shows that the particle insertion in the $\kappa \to 0$ corresponds to the following observable of the abelian case:

$$\mathcal{O}_e(m_e) = \kappa \frac{\delta(l_e)}{l_e^2} \times \frac{\pi}{2m}\delta\left(|u_e| - m_e\right). \tag{55}$$

Finally, putting everything together, the QFT limit of the Ponzano-Regge amplitude with a particle graph $\{\Gamma, m_e\}$ is simply equal to the amplitude for particle insertions in the abelian partition function and reads:

$$\begin{aligned} Z_\Delta^{(ab)}(\Gamma, \{m_e\}) &= \lim_{\kappa\to 0} \kappa^{|e_\Gamma|} \mathcal{Z}_\Gamma(\theta_e) \\ &= \int \prod_{e^*} d\vec{u}_{e^*} \prod_{e\in\Gamma} \frac{\delta\left(|u_e| - m_e\right)}{4\pi m_e} \prod_{e\notin\Gamma} \delta(u_e). \end{aligned} \tag{56}$$

where $|e_\Gamma|$ is the number of edges in Γ.

This shows that the QFT limit $\kappa \to 0$ of the Ponzano-Regge amplitude reproduces the usual classical Feynman graph evaluation (with Hadamard propagators), which was previously shown to be equal to the amplitudes of the abelian $\mathbb{R}^3$ spin foam model of section 2. The next step is to expand the non-abelian spin foam amplitudes beyond the first order in κ in order to understand the structure of the Feynman graph evaluation in the non-abelian context or, in other words, extract the quantum gravity correction to the classical QFT Feynman graph evaluations.

4.2 Expansion in G: Non-Commutative Geometry and Star Product

4.2.1 Feynman evaluation of spherical graphs

We start with the Ponzano-Regge spin foam amplitude for a particle graph written in section 3. We first deal with the case of a spherical graph Γ. Applying the (topology) duality transform described in section 2, we can write this amplitude exactly as a Feynman graph evaluation similar to (6).

More precisely, if Γ is a spherical graph $\Sigma_g \sim \mathcal{S}^2$, there is no cycle variables a_i, b_i to integrate over and the spin foam amplitude (39) simplifies down to (31). Then let's denote $G_e = g_{t(e)} g_{s(e)}^{-1}$ the group element associated to each edge $e \in \Gamma$, where $t(e), s(e)$ label the two triangles bounding e. It is clear that the product of such G_e variables around any given vertex $v \in \Gamma$ is constrained to be the identity. And in the end, we obtain the following Feynman graph evaluation:

$$\mathcal{I}(\Gamma,\theta) \equiv \mathcal{Z}_{\mathcal{S}^3}(\Gamma_\theta) = \int_{\mathrm{SU}(2)^E} \prod_{e\in\Gamma} dG_e\, \Delta(G_e)\delta_{\theta_e}(G_e) \prod_v \delta(G_v), \tag{57}$$

with

$$G_v = \overrightarrow{\prod_{e\supset v}} G_e^{\epsilon_v(e)},$$

where $\epsilon_v(e) = \pm$ whether v is the target or source vertex of the edge e. Now one can re-express the δ function on the group $SO(3)$ as[8]:

$$\delta(G) = \frac{1}{8\pi\kappa^3} \int_{\mathfrak{su}(2)} d^3X\, e^{\frac{i}{2\kappa} Tr(Xg)}. \tag{58}$$

Therefore the non-abelian Feynman evaluation reads:

$$\mathcal{I}(\Gamma,\theta) = \int \prod_v \frac{d^3X_v}{8\pi\kappa^3} \int \prod_e dG_e\, \Delta(G_e)\delta_{\theta_e}(G_e) \prod_v e^{\frac{i}{2\kappa} Tr(X_v G_v)}. \tag{59}$$

If the product $G_v = \prod_{e\in\partial v} G_e$ was abelian, we could expand the exponent and reorganize the product over vertices into a product over edges just like the usual Feynman graph evaluation (5). Nevertheless, we know that the product is actually abelian at the first order in κ, so that we can do a perturbative expansion in κ and compute deviations from the classical Feynman diagram expressions.

Denoting $2i\kappa\vec{P} = Tr(g\vec{\sigma})$ the projection of the group element g on the Lie algebra, the product of n group elements $g_1, .., g_n$ admits a simple expansion in terms of κ:

$$\prod_i g_i \;=\; 1 + i\kappa \sum_i \vec{P}_i\cdot\vec{\sigma} - \frac{\kappa^2}{2}\left|\sum_i \vec{P}_i\right|^2 - i\kappa^2 \left(\sum_{i<j} \vec{P}_i \wedge \vec{P}_j\right)\cdot\vec{\sigma} + \dots \tag{60}$$

[8]This formula gives us the delta function on $SO(3) = SU(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the identification $g = -g$. In the following we restrict to the case where all group elements are $SO(3)$ group elements for simplicity. As pointed out in [3] if we want to construct the delta function on $SU(2)$ the correct formula is:

$$\delta(G) = \frac{1}{8\pi} \int_{\mathfrak{su}(2)} d^3X\, e^{\frac{i}{2} Tr(Xg)}(1+\epsilon(g)),$$

where $\epsilon(g)$ is the sign of $\cos\theta$ with $g = \cos\theta + i\sin\theta = \hat{n}\cdot\vec{\sigma}$.

Then we can expand the exponent in the graph amplitude:

$$\frac{i}{2\kappa}Tr(X_vG_v) = -i\vec{X}_v.\left(\sum_{e\in\partial v}\epsilon_v(e)\vec{P}_e + \kappa\epsilon_v(e)\epsilon_v(e')\sum_{e<e'\in\partial v}\vec{P}_e\wedge\vec{P}_{e'} + \dots\right), \tag{61}$$

where $X_v = i\vec{X}_v.\vec{\sigma}$. So the full Feynman evaluation reads:

$$\mathcal{I}(\Gamma,m) = \int\prod_v\frac{d^3X_v}{8\pi\kappa^3}\int\prod_e dG_e\,\Delta(G_e)\delta_{\kappa m_e}(G_e)\prod_v e^{-i\vec{X}_v.\left(\sum_e\epsilon_v(e)\vec{P}_e+\kappa\epsilon_v(e)\epsilon_v(e')\sum_{e,e'}\vec{P}_e\wedge\vec{P}_{e'}+\dots\right)}$$

$$= \int\prod_v\frac{d^3X_v}{8\pi\kappa^3}\int\prod_e dG_e\,\Delta(G_e)\delta_{\kappa m_e}(G_e)\prod_v\left(1 - i\kappa\epsilon_v(e)\epsilon_v(e')\vec{X}_v.\sum_{e,e'}\vec{P}_e\wedge\vec{P}_{e'} + \dots\right)e^{-i\vec{X}_v.\sum_e\epsilon_v(e)\vec{P}_e} \tag{62}$$

We also need to expand the measure:

$$\int\prod_e dG_e\,\Delta(G_e)\delta_{\kappa m_e}(G_e)\,\varphi(G_e) \equiv \int\prod_e\frac{\kappa d^3\vec{P}_e}{2\pi}\,\delta\left(|\vec{P}_e|^2 - \left(\frac{\sin m_e\kappa}{\kappa}\right)^2\right)\varphi(\vec{P}_e),$$

in order to express everything in terms of the moment vectors $\vec{P}_e$ and get the full perturbative expansion in κ. This leads to a simple renormalisation of the mass $m \to \frac{\sin\kappa m}{\kappa}$, so that the measure doesn't contribute at first order in κ. If one introduces a source $\vec{J}_e$ for the variables $\vec{P}_e$, one can generate the polynomial terms in $\vec{P}$ through some derivative with respect to the source J:

$$\mathcal{I}(\Gamma,m) = \int\prod_v\frac{d^3X_v}{8\pi\kappa^3}\left(1 + i\kappa\vec{X}_v.\sum_{e,e'\in\partial v}\epsilon_v(e)\epsilon_v(e')\frac{\delta}{\delta\vec{J}_e}\wedge\frac{\delta}{\delta\vec{J}_{e'}} + \dots\right).I(\Gamma,m_e,\vec{X}_v,\vec{J}_e)\Bigg|_{J=0}, \tag{63}$$

where $I(\Gamma,m,\vec{X}_v,\vec{J})$ is a normal (abelian) Feynman graph evaluation:

$$I(\Gamma,m,\vec{X}_v,\vec{J}) = \int\prod_e\frac{\kappa d^3\vec{P}_e}{2\pi}\,\delta\left(|\vec{P}_e|^2 - \frac{\sin^2 m_e\kappa}{\kappa^2}\right)\varphi(\vec{P}_e)\prod_e e^{-i\kappa\vec{P}_e.(\vec{X}_{t(e)}-\vec{X}_{s(e)})-i\vec{J}_e.\vec{P}_e}. \tag{64}$$

Therefore we see that the Feynman graph evaluation for particles in 3d quantum gravity can be expressed as a perturbative expansion in κ with operators acting on a classical Feynman graph evaluation.

At first order the operator to insert is a sum of operator acting at the vertices of the Feynman diagram which modify how the particles interact. At higher orders the measure will also contribute and modify the propagator of the theory.

One can compute the higher order correction and this perturbative expansion looks quite cumbersome at first sight. However we are now going to see how the full perturbative expansion can be simplified if one re-express the previous formulas in terms of a *star product*, which renders explicit the non-commutative structure of the theory and scattering amplitudes.

4.2.2 Star Product and Non-Commutative Field Theory

Starting from the non-abelian Feynman evaluation (59), one would like to split the exponentials $\exp(\frac{i}{2}Tr(X_v\prod_e G_e))$ in order to re-organize all the exponentials and write the evaluation as a

standard Feynman amplitude (as in section 2). It is thus natural to introduce a notion of $\star$-product on functions of $X \in \mathfrak{su}(2) \sim \mathbb{R}^3$ defined through:

$$e^{\frac{i}{2\kappa}Tr(Xg_1)} \star e^{\frac{i}{2\kappa}Tr(Xg_2)} \equiv e^{\frac{i}{2\kappa}Tr(Xg_1g_2)}. \tag{65}$$

This leads to introduce the following Fourier transform between the Lie algebra $\mathfrak{su}(2)$ and the Lie group SO(3):

$$f(X) = \int dg\, e^{\frac{i}{2\kappa}Tr(Xg)} \tilde{f}(g). \tag{66}$$

This $\star$-product is non-commutative but still associative. Moreover it can be seen as equivalent to the convolution product on the group:

$$\widetilde{(\phi \star \psi)} = \widetilde{\phi} \circ_G \widetilde{\psi}, \qquad \text{with} \quad \widetilde{\phi} \circ_G \widetilde{\psi}(g) = \int dh\, \widetilde{\phi}(gh^{-1})\widetilde{\psi}(h). \tag{67}$$

We can also define the star convolution product on $\mathbb{R}^3$ which is dual to the usual product of functions on the group.

$$\widetilde{(\phi \circ_\star \psi)} = \widetilde{\phi}\widetilde{\psi}, \qquad \text{with} \quad \phi \circ_\star \psi(X) \equiv \int_{\mathbb{R}^3} d^3X \phi(X-Y) \underset{Y}{\star} \psi(Y). \tag{68}$$

A natural set of functions to study are the Fourier transforms f_j of the character $\chi_j(g)$. It is easy to derive that:

$$f_j \star f_k = \frac{\delta_{jk}}{d_j} f_j, \tag{69}$$

$$f_j \circ_\star f_k = \sum_{l=|j-k|}^{j+k} f_l. \tag{70}$$

More generally, if we denote $D^j_{ab}(g)$ the matrix elements in the spin j representation, we can define:

$$f^j_{ab}(X) = \int dg\, e^{\frac{i}{2\kappa}Tr(Xg)} D^j_{ab}(g),$$

we can check that:

$$\begin{aligned} f^j_{ab} \star f^k_{cd} &= \frac{\delta_{jk}}{d_j} \delta_{bc} f^j_{ad} \\ f^j_{ab} \circ_\star \left(f^k_{cd}\right)^\dagger &= \sum_{l,e,f} C^{jkl}_{ace} C^{jkl}_{bdf} f^l_{ef}, \end{aligned} \tag{71}$$

where we have defined $\widetilde{f^{k\,\dagger}_{cd}}(g) = D^k_{cd}(-g^{-1})$ and the Clebsh-Gordan coefficient C^{jkl} of the recoupling theory for representations of SU(2). This shows that the f_j's project onto the j representation for the $\star$-product:

$$f_j \star f^k_{cd} = \frac{\delta_{jk}}{d_j} f^j_{cd}.$$

In particular, f_0 is a projector:

$$f_0 \star f^k_{cd} = \delta_{0k} f_0, \tag{72}$$

and plays the role of the usual $\delta(x)$ distribution. In order to understand the behavior of the functions f_j, it is useful to look at the link between this algebra/group Fourier transform and the standard Fourier transform on $\mathbb{R}^3$. We can express SO(3) group elements in terms of their projections on the Lie algebra

$$g = \sqrt{1-\kappa^2|\vec{P}|^2} + i\kappa\vec{P}.\vec{\sigma} \tag{73}$$

A function f on SO(3) can be equivalently seen as a function on the 3-ball $B^3_\kappa = \{|\vec{P}| \le \kappa^{-1}\}$ [9]. We call this space of function $\widetilde{\mathcal{B}}_\kappa(\mathbb{R}^3)$. This is our Fourier space: by construction no momenta in this space takes a value larger than κ^{-1}. Our group Fourier transform is related to the usual fourier transform by:

$$f(X) = \frac{\kappa^3}{\pi^2}\int_{B^3_\kappa} \frac{d^3\vec{P}}{\sqrt{1-\kappa^2|\vec{P}|^2}}\, e^{-i\vec{X}.\vec{P}}\tilde{f}(P). \tag{74}$$

The functions f_j can be seen to be related to the Bessel functions of the first kind. The important feature is that f_0 has a non-zero width. Therefore, as f_0 plays the role of $\delta(X)$ for the $\star$-product, this means we have access only to a finite resolution: the $\star$-product tells us of a minimal length scale (maximal resolution) on the spacetime (X sector). This is simply due to the fact that the momentum space (P sector) is bounded.

In order to describe the functional space dual to $\widetilde{\mathcal{B}}_\kappa(\mathbb{R}^3)$, we introduce the following kernel:

$$G(X,Y) = \int_{B^3_\kappa} \frac{d^3\vec{P}}{(2\pi)^3} e^{i(\vec{X}-\vec{Y})\cdot\vec{P}}. \tag{75}$$

This is a projector as we can check that $\int d^3X\, G(X,Y)G(Y,Z) = G(X,Z)$. It is now straightforward to show that if $f(X)$ is the group Fourier transform of a function $\tilde{f}$ then $G(f) = f$. Moreover any function in the image of G has a Fourier transform with support in B^3_κ, therefore the image of $L^2(\mathbb{R}^3)$ by this projector, denoted $\mathcal{B}_\kappa(\mathbb{R}^3)$, is isomorphic to $\widetilde{\mathcal{B}}_\kappa(\mathbb{R}^3)$ by the group Fourier transform (74). This Fourier transform is in fact an isometry if we provide $\widetilde{\mathcal{B}}_\kappa(\mathbb{R}^3)$ and $\mathcal{B}_\kappa(\mathbb{R}^3)$ with the following norms:

$$\begin{aligned} ||\phi||^2_{\mathcal{B}_\kappa} &\equiv \int_{\mathbb{R}^3} \frac{d^3X}{8\pi\kappa^3}(\phi\star\bar{\phi})(X), \\ ||\widetilde{\phi}||^2_{\widetilde{\mathcal{B}}_\kappa} &\equiv \frac{\kappa^3}{\pi^2}\int_{\widetilde{\mathcal{B}}_\kappa} \frac{d^3P}{\sqrt{1-\kappa^2|\vec{P}|^2}}|\widetilde{\phi}(\vec{P})|^2. \end{aligned} \tag{76}$$

With the tool of this star-product, the Feynman evaluation (59) reads:

$$\mathcal{I}(\Gamma,\theta) = \int\prod_v \frac{d^3X_v}{8\pi\kappa^3}\int\prod_e dG_e\,\Delta(\kappa m)\delta_{\kappa m_e}(G_e)\prod_v\left(\bigstar_{e\in\partial v} e^{\frac{i}{2\kappa}Tr(X_v G_e^{\epsilon_v(e)})}\right). \tag{77}$$

[9] If f is continuous at the identity, it satisfies the additional condition $f(\kappa^{-1}\vec{n})$ is independent of $\vec{n}$ a unit vector. we will not impose this condition in general

When written in terms of momenta [10]

$$\mathcal{I}(\Gamma,\theta)=\left(\prod_e \frac{\pi\cos\kappa m_e}{2\kappa^2}\right)\int\prod_v \frac{d^3X_v}{8\pi\kappa^3}\int\prod_e \frac{\kappa^3\, d^3\vec{P}_e}{\pi^2\sqrt{1-\kappa^2|P|^2}}\,\delta\left(|\vec{P}_e|^2-\frac{\sin^2 m_e\kappa}{\kappa^2}\right)\prod_v\left(\bigstar_{e\in\partial v} e^{i\,\epsilon_v(e)\vec{X}_v\cdot\vec{P}_e}\right). \tag{81}$$

So far the propagator which enters this amplitude is the Hadamard propagator $\delta\left(|\vec{P}_e|^2-\frac{\sin^2 m_e\kappa}{\kappa^2}\right)$. We can write this propagator as a proper time integral:

$$\delta\left(|\vec{P}_e|^2-\frac{\sin^2 m_e\kappa}{\kappa^2}\right)=\int_{-\infty}^{+\infty}\frac{dT}{2\pi}\,e^{iT\left(|\vec{P}_e|^2-\frac{\sin^2 m_e\kappa}{\kappa^2}\right)}.$$

If one restricts the integral to be over positive proper time $T\in\mathbb{R}_+$, one obtains the usual Feynman propagator with a renormalized mass. Using this Feynman propagator, the spin foam amplitude reads

$$\mathcal{I}_F(\Gamma,\theta)=\prod_e\left(\frac{\cos\kappa m_e}{4\kappa^2}\right)\int\prod_v\frac{d^3X_v}{8\pi\kappa^3}\int\prod_e dg_e\,\frac{i}{|\vec{P}_e|^2-\frac{\sin^2 m_e\kappa}{\kappa^2}+i\epsilon}\prod_v\left(\bigstar_{e\in\partial v} e^{i\,\epsilon_v(e)\vec{X}_v\cdot\vec{P}_e}\right). \tag{82}$$

with $\vec{P}_e\equiv\vec{P}(g_e)$.

In the next section, we will show that these amplitudes are truly the Feynman graph evaluations of a non-commutative field theory for a κ-deformed Poincaré group. Such field theory then acquires the interpretation of an effective theory for matter propagating in the 3d quantum geometry.

4.2.3 Non-spherical Graphs and non-trivial Braiding

Beyond the case of spherical graphs, dealing with graphs with an surface embedding of non-trivial topology $g\neq 0$ is more subtle. It will lead us to the notion of braided Feynman diagrams for a non-commutative field theory.

Starting with the graph amplitude (39) and defining the edge group elements $H_e=g_{t(e)}a_e^{\epsilon(e)}g_{s(e)}^{-1}b_e^{\epsilon(e)}$, it is obvious that the product of such H_e variables around a vertex $v\in\Gamma$ will not generally be the identity. First, let us notice an ambiguity in the definition of the group element associated to an edge. Indeed the distribution δ_θ is invariant under conjugation $\delta_\theta(\cdot)=\delta_\theta(k\cdot k^{-1})$, so that we can in general choose some arbitrary variables k_e and use the group elements $G_e=k_eH_ek_e^{-1}$. To fix this ambiguity and build rigorously the group variables G_e, one uses a methods similar to the gauge-fixing techniques used in [10, 3, 13]. To follow the procedure, it is easier to work on the dual graph to Γ. One chooses a dual vertex $\bar{v}_0$ of reference (an initial face f_0) and a maximal tree T on the dual graph. For any dual vertex $\bar{v}$, there exists a unique path $\mathcal{P}(\bar{v}_0\to\bar{v})$ running from $\bar{v}_0$ to $\bar{v}$ along the tree T. Then one can define the ordered product $k_{\bar{v}}$ of group elements $g_{\bar{w}}$

[10] The normalizations relating g and P are given by

$$\int dg = \frac{\kappa^3}{\pi^2}\int_{B_\kappa}\frac{d^3\vec{P}}{\sqrt{1-\kappa^2|P|^2}}, \tag{78}$$

$$\delta_{\kappa m}(g) = \frac{\pi}{2\kappa^2}\frac{\cos\kappa m}{\sin\kappa m}\delta\left(|\vec{P}_e|^2-\frac{\sin^2\kappa m_e}{\kappa^2}\right), \tag{79}$$

$$\Delta(\kappa m) = \sin\kappa m. \tag{80}$$

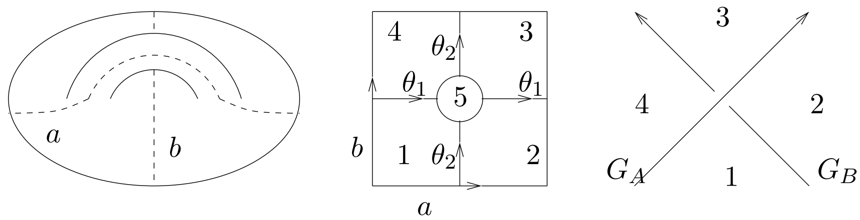


Figure 1: Case of a crossing: we lift the crossing by embedding it into a handle, we triangulate the corresponding punctured torus and we derive the braiding rule in the non-abelian Feynman evaluation.

associated to the dual vertices along the path $\mathcal{P}(\bar{v}_0 \to \bar{v})$. One should moreover include all the cycles a_i, b_i that the path crosses: if the dual edge between the consecutive dual vertices $\bar{w}$ and $\bar{w}'$ intersects a cycle (i.e if the corresponding edge belongs to the cycle), then one includes the group element corresponding to that cycle in the ordered product $k_{\bar{v}}$. Then one would define the edge group element $G_e = k_{t(e)} H_e k_{s(e)}^{-1}$, which is simply the ordered product of face group elements g_f and cycle variables around the edge e but starting at a fixed reference face f_0. Nevertheless, the (ordered) product of the G_e's around all vertices is still not trivial and the situation is more subtle than a simple choice of origin on the dual complex.

The moot point is at the level of crossings: when two edges of the graph cross each other (on a 2d projection of the graph or more precisely when embedding the graph on the sphere), we have a non-trivial braiding of the edge group elements, which forces us to associate two group elements to each edge, one at the source vertex and one at the target vertex.

First Γ being a non-spherical graph means that we cannot draw it on a sphere without crossings. In order to compute the amplitude we need to choose a surface on which this graph is drawn without crossing. There is an arbitrariness on the choice of this surface of course. Our amplitude being topological, it doesn't matter which surface we choose and we are free to choose the most convenient one. The surface which we use consists of adding one handle to the sphere for each crossing: the upper edge goes through the handle whose meridian is a cycle a and the lower edge goes below the arch of the crossing as shown in fig.1. This means that starting from a projection of our diagram on the sphere we cut a small disk around each crossing of the projection and glue back a punctured torus (with one hole). Topological invariance insures that the amplitude doesn't depend on the choice of projection of our graph.

Then let us focus on one crossing. We choose the simplest cell decomposition of the punctured torus as on fig.1. There is 4 faces to which we assign group elements $g_1, .., g_4$. In this diagram there are 8 edges. Each edge lead to a potential constraint. 4 of them are mass constraints, only two (one per edge) are relevant. Among the other 4 constraints, one can be gauged away (Lets say the one associated with the edge bounding the face 2 and 3). Then the spin amplitude (39) for the concerned faces and edges around the crossing reads:

$$\delta_{\theta_1}(g_1 g_4^{-1})\delta_{\theta_2}(g_1 g_2^{-1})\delta(g_2 g_1^{-1} b)\delta(g_3 g_4^{-1} b)\delta(g_4 a g_1^{-1})$$

Integrating over the cycle group elements a, b leaves a single constraint $\delta(g_1 g_2^{-1} g_3 g_4^{-1})$ plus the mass constraints. We then introduce the group elements associated to the two edges at each end:

$$G_A^s = g_4 g_1^{-1}, G_A^t = g_3 g_2^{-1}, G_B^s = g_1 g_2^{-1}, G_B^t = g_4 g_3^{-1}.$$

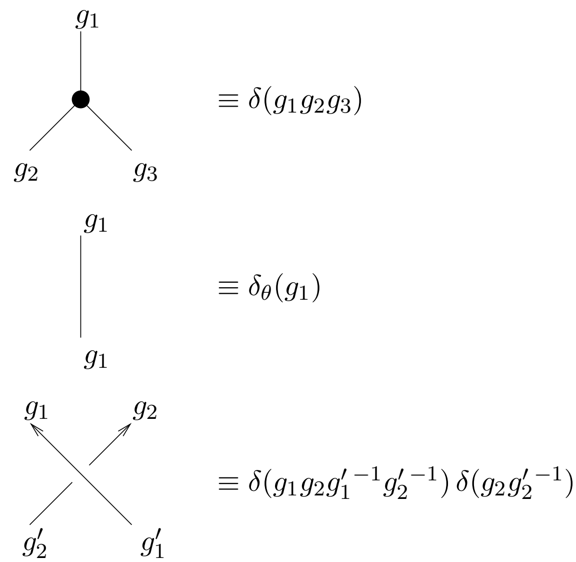


Figure 2: Feynman rules for particles propagation in the Ponzano-Regge model.

They satisfy the constraints $G^s_A G^s_B (G^t_A)^{-1} (G^t_B)^{-1} = 1$. In terms of these edge variables, the spin foam amplitude reads:

$$\delta_{\theta_1}(G^s_A) \delta_{\theta_2}(G^s_B) \delta(G^s_A G^s_B (G^t_A)^{-1} (G^t_B)^{-1}) \delta(G^s_B (G^t_B)^{-1}). \tag{83}$$

The important thing to notice is that we do not have a constraint $\delta(G^s_A (G^t_A)^{-1})$.

We can now fully specify our Feynman rules. We first assign two group elements to each edge $G_{s(e)}, G_{t(e)}$. For each vertex we put a weight $\delta(\prod_{e \supset v} G_{s(e)})$, for each edge with no crossing we put a weight $\delta_{\theta_e}(G_{s(e)}) \delta(G_{s(e)} (G_{t(e)})^{-1})$ and for each crossing we add $\delta(G_{s(e')} G_{s(e)} (G_{t(e')})^{-1} (G_{t(e)})^{-1}) \delta(G_{s(e)} (G_{t(e)})^{-1})$ where e' over-crosses e. This is summarized in fig.2. We show in the next section that these Feynman rules can be derived exactly from a non-commutative field theory based on the κ-deformed Poincaré group.

These Feynman rules can be understood as a Reshetikhin-Turaev evaluation of a colored graph constructed with the quantum group $\mathcal{D}(SU(2))$ (which is a κ-deformation of the Poincaré group) [4]. They involve a non-trivial braiding factor for each crossing. We can do the same trick as previously and promote the Hadamard amplitudes to Feynman amplitudes by taking only the integration over positive proper time in the propagator and we call the resulting amplitudes $\mathcal{I}_F(\Gamma, \theta)$.

5 A Non-Commutative Braided Field theory

5.1 Non-Commutative Field Theory as Effective Quantum Gravity

We now introduce a non-commutative field theory based on the previous $\star$-product, and we show it leads to a sum over graphs of the particle insertion amplitudes obtained from the spin foam model. It has the natural interpretation of an effective field theory describing the dynamics of matter in quantum gravity after integration of the gravitational degrees of freedom.

Let us now consider the graph evaluations $\mathcal{I}_F(\Gamma,\theta)$ given by the equation (82). We consider the case where we have particle of only one species so all masses are equal, $m_e \equiv m$. Having different masses would only require to introduce more fields and would not modify the overall picture in any way. We define a normalized amplitude[11] $\widetilde{\mathcal{I}}_F(\Gamma,\theta) = \left(\frac{4\kappa}{\cos\kappa m}\right)^{|e_\Gamma|}\mathcal{I}_F(\Gamma,\theta)$ where $|e_\Gamma|$ is the number of edges of Γ. Let us now consider the sum over planar and trivalent graphs:

$$\sum_{\Gamma \text{planar,trivalent}} \frac{\lambda^{|v_\Gamma|}}{S_\Gamma}\widetilde{\mathcal{I}}_F(\Gamma,\kappa m) \tag{84}$$

over trivalent planar diagrams (we will include non-planar diagrams in the next section) where λ is a coupling constant, $|v_\Gamma|$ is the number of vertices of Γ and S_Γ is the symmetry factor of the graph. Remarkably, this sum can be obtained from the perturbative expansion of a non-commutative field theory given explicitly by:

$$S = \frac{1}{8\pi\kappa^3}\int d^3x \left[\frac{1}{2}(\partial_i\phi \star \partial_i\phi)(x)^2 - \frac{1}{2}\frac{\sin^2 m\kappa}{\kappa^2}(\phi\star\phi)(x)^2 + \frac{\lambda}{3!}(\phi\star\phi\star\phi)(x)\right] \tag{85}$$

where the field ϕ is in $\widetilde{\mathcal{B}}_\kappa(\mathbb{R}^3)$. Its moment has support in the ball of radius κ^{-1}. We can write this action in momentum space

$$S(\phi) = \frac{1}{2}\int dg\,\left(P^2(g) - \frac{\sin^2\kappa m}{\kappa^2}\right)\widetilde{\phi}(g)\widetilde{\phi}(g^{-1}) + \frac{\lambda}{3!}\int dg_1 dg_2 dg_3\,\delta(g_1g_2g_3)\,\widetilde{\phi}(g_1)\widetilde{\phi}(g_2)\widetilde{\phi}(g_3), \tag{86}$$

from which it is straightforward to read the Feynman rules and show our statement.

It is interesting to write down the interaction term in terms of the momenta $P(g)$. The momentum addition rule becomes non-linear, in order to preserve the condition that momenta is bounded, explicitly:

$$\vec{P}_1 \oplus \vec{P}_2 = \sqrt{1-P_2^2}\,\vec{P}_1 + \sqrt{1-P_1^2}\,\vec{P}_2 - \vec{P}_1\wedge\vec{P}_2. \tag{87}$$

Therefore the momentum conservation at the interaction vertex reads:

$$P_1\oplus P_2\oplus P_3 = 0 = P_1+P_2+P_3 - \kappa(P_1\wedge P_2 + P_2\wedge P_3 + P_1\wedge P_3) + \mathcal{O}(\kappa^2). \tag{88}$$

From this identity, it appears that the momenta is non-linearly conserved, and the non-conservation is stronger when the momenta are non-collinear. The natural interpretation is that part of the energy involved in the collision process is absorbed by the gravitational field, this effect prevents any energy involved in a collision process to be larger than the Planck energy. This phenomena is simply telling us that when we have a high momentum transfer involve in a particle process, one can no longer ignore gravitational effects which are going to modify how the energy is transferred.

The non-commutative field theory action is symmetric under a κ-deformed action of the Poincaré group. If we denote by Λ the generators of Lorentz transformations and by $T_{\vec{a}}$ the generators of translations, it appears that the action of these generators on one-particle states is undeformed:

$$\Lambda\cdot\tilde{\phi}(g) = \tilde{\phi}(\Lambda g\Lambda^{-1}) = \tilde{\phi}(\Lambda\cdot P(g)), \tag{89}$$

$$T_{\vec{a}}\cdot\tilde{\phi}(g) = e^{i\vec{P}(g)\cdot\vec{a}}\tilde{\phi}(g). \tag{90}$$

[11] From the spin foam point of view each amplitude correspond to a manifold of different topology. The spin foam approach does not specify how one should relatively weight amplitude of different topology and we have to make a natural choice.

Since there seems to be a certain amount of confusion in the literature on this subject, we would like to emphasize that it is impossible to deform non-trivially the action of the Poincaré group on one-particle states. Indeed, it is well known that the cohomology group of the Poincaré group is trivial: any deformation of the Poincaré group which is connected to the identity (i.e. which depend continuously on a parameter κ) can be undone by a non-linear redefinition of the algebra generators. More precisely, let us start from any κ-deformation (which preserve associativity or Jacobi identity) of the Poincaré algebra, that is we have Poincaré generators $J_{\mu\nu}, P_\mu$ with deformed commutation relations such that when $\kappa = 0$ we recover the usual algebra. Then we can always choose new generators $\tilde{J}_{\mu\nu}, \tilde{P}_\mu$ which are non-linear functions the original generators and κ such that the new generators satisfy the undeformed Poincaré algebra. This choice of generators is referred as the classical basis. This is nicely exemplified in [5] for the 3+1 κ-Poincaré algebra and in [12] for the 2+1 κ-Poincaré algebra. The open problem in this context is whether physics does depend on the choice of basis or not and if so which one is preferred. It seems that the choice of basis matters, at first sight, since each basis correspond to a choice of what should be used as an operational notion of energy[14], and for instance the dispersion relation is dependent on the basis choice. Whether this is a true physical dependance or not is still however a matter of debate [15].

It has already been noticed that particles in 3d quantum gravity should be described by a Deformed/Doubly Special Relativity (DSR) [3, 12]. What we learn from the present analysis is that 2+1 gravity chooses for us a preferred basis which is a classical basis (this fact was already noticed in [12]). Note however that if $J_{\mu\nu}, P_\mu$ form a classical basis, then any redefinition of the form $\tilde{J}_{\mu\nu} = J_{\mu\nu}, \tilde{P}_\mu = P_\mu f(\kappa|P|)$, with an arbitrary function f, still yields a classical basis. So 2+1 gravity chooses one particular classical basis[12]. This chosen one, singled out by gravity, is consistent with the DSR principle since it is not possible in this basis to have an energy which exceed the Planck energy.

In a classical basis, the non-trivial deformation of the Poincaré group appears at the level of multi-particle states. Indeed, the action of the Poincaré group on two-particle state is given by

$$\begin{aligned} \Lambda \cdot \tilde{\phi}(P_1)\tilde{\phi}(P_2) &= \tilde{\phi}(\Lambda \cdot P_1)\tilde{\phi}(\Lambda \cdot P_2), && (91) \\ T_{\vec{a}} \cdot \tilde{\phi}(P_1)\tilde{\phi}(P_2) &= e^{i\vec{P}_1 \oplus \vec{P}_2 \cdot \vec{a}}\tilde{\phi}(P_1)\tilde{\phi}(P_2). && (92) \end{aligned}$$

The action of the Lorentz generator is undeformed[13] however the action of the translations is modified in a non-trivial fashion reflecting the fact that the spacetime has become non-commutative. With these rules it is clear that our field action is κ-Poincaré invariant. In algebraic language this means that the symmetry algebra is promoted to a non-cocommutative Hopf algebra. In the classical basis, the only non-trivial coproduct is[14]

$$\Delta(P_i) = \sqrt{1 - P^2} \otimes P_i + P_i \otimes \sqrt{1 - P^2} - \epsilon_{ijk} P^j \otimes P^k. \tag{93}$$

This shows that the effective theory describing the dynamics of second quantized particles in the presence of gravity can be effectively described, after integration over the gravity degrees of freedom as an explicit non-commutative field theory which respects DSR principles.

[12] If we denote p the standard classical basis and P the one appearing in our context the relation is given by $P^a = \frac{\sin \kappa|p|}{\kappa|p|} p^a$.

[13] This is different from the κ-deformation of the 3+1 Poincaré algebra.

[14] This is exactly the Hopf algebra structure of the κ-deformation of the 2+1 Poincaré algebra, or equivalently of the quantum double $\mathcal{D}(\mathrm{SU}(2))$, which appears in the quantization of 2+1 gravity [4, 17]. The infinitesimal deformation

In fact we will now see by looking at the non-planar graphs that the proper effective description is in term of a braided non-commutative field theory which takes into account the non-trivial statistics induced by gravity. Also, the Euclidean gravity amplitudes which we have discussed so far are expressed in terms of Hadamard propagator and we had to extract the Feynman propagator from them, in order to make contact with non-commutative field theory. We will argue in the next section that the Lorentzian gravity spin foam amplitudes should have a formulation in which gravity amplitudes are given in terms of causal propagators.

5.2 Non-Commutativity and Braided Feynman Diagrams

The question is now whether the non-commutative field theory introduced above reproduces the quantum gravity amplitude in the non-planar case. The computation of Feynman amplitudes in the planar and non-planar case differs in the sense that in the non-planar case we have to commute the Fourier modes of the field before doing any Wick contraction. For instance if we compute using Wick theorem the following vacuum expectation value, we get

$$\begin{aligned}\langle\widetilde{\phi}(P_1)\widetilde{\phi}(P_2)\widetilde{\phi}(P_3)\widetilde{\phi}(P_4)\rangle &= \langle\widetilde{\phi}(P_1)\widetilde{\phi}(P_2)\rangle\,\langle\widetilde{\phi}(P_3)\widetilde{\phi}(P_4)\rangle + \langle\widetilde{\phi}(P_1)\widetilde{\phi}(P_4)\rangle\,\langle\widetilde{\phi}(P_2)\widetilde{\phi}(P_3)\rangle \\ &\quad + \langle\widetilde{\phi}(P_1)\widetilde{\phi}(P_3)\rangle\,\langle\widetilde{\phi}(P_2)\widetilde{\phi}(P_4)\rangle. \end{aligned} \tag{94}$$

If we draw the corresponding Feynman diagrams, putting the momenta ordered on a line we see that the first two contractions are planar whereas the third one contain a crossing, the crossing is due to the fact that we have to exchange $\widetilde{\phi}(P_2)$ and $\widetilde{\phi}(P_3)$ before making the Wick contraction. In order to compute the non-planar Feynman diagrams we therefore have to specify the rules for commuting the modes $\widetilde{\phi}(P)$, that is specify the *statistics* of our particles. In standard commutative local field theory, we know that only two statistics are usually possible bosons or fermions (except in $2+1$ dimensions). What about non-commutative field theory? It seems that this is terra incognita since the usual spin statistics theorem cannot be applied (except for some particular examples in usual non-commutative field theory with no space time non commutativity [16]). We now argue that once we have fixed the star product and the duality between space and time there is a natural way to specify the statistics of our field. Let us look at the product of two identical fields:

$$\phi\star\phi\,(x) = \int dg_1 dg_2\, e^{\frac{i}{2\kappa}\,\mathrm{tr}(xg_1g_2)}\widetilde{\phi}(g_1)\widetilde{\phi}(g_2), \tag{95}$$

We can move $\tilde{\phi}(g_2)$ to the left by making the following change of variables $g_1 \to g_2$ and $g_2 \to g_2^{-1}g_1g_2$, the star product reads

$$\phi\star\phi\,(x) = \int dg_1 dg_2\, e^{\frac{i}{2\kappa}\,\mathrm{tr}(xg_1g_2)}\widetilde{\phi}(g_2)\widetilde{\phi}(g_2^{-1}g_1g_2), \tag{96}$$

This suggest that the proper way to read the statistics of our non commutative field is to assume that they satisfy the commutation relation:

$$\tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2) \tag{97}$$

of the algebra is simply given by: $\delta(J_i)=0$, $\delta(P_i)=\epsilon_{ijk}P^j\otimes P^k$. The κ-deformation of the Poincaré algebra in 3+1 dimensions is actually different and usually involves singling out the time direction and also deforming the Lorentz generators.

In our case, we can check that this commutation relation is exactly the one coming from the braiding of two particles coupled to quantum gravity. This braiding was computed in the spin foam model [3] and is encoded into a braiding matrix

$$R \cdot \tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2). \tag{98}$$

This is the R matrix of the κ-deformation of the Poincaré group [18]. We see that the non-trivial statistics imposed by the study of our non-commutative field theory is related to the braiding of particles in 3 spacetime dimensions[15] Such field theory with non-trivial braided statistics are usually simply called braided non-commutative field theory and they were first introduced in [19].

If one uses the non-trivial statistic (97), it becomes an easy exercise to check that the partition function of our non-commutative field theory reproduces the sum over all quantum gravity amplitudes with insertions of Feynman propagators.

6 On the classical limit of the Turaev-Viro model

6.1 The many limits of the Turaev-Viro model

We have dealt so far with the case of a non zero cosmological constant Λ. When $\Lambda \neq 0$ the quantum gravity amplitudes are known to be given by the Turaev-Viro model instead of the Ponzano-Regge model. We want to show in this section that the zero gravity limit of Turaev-Viro reproduces computation of Feynman diagram with insertion of the Hadamard propagator on $\mathcal{S}^3$. We also present a modification of the Turaev-Viro model which reproduces, in the no gravity limit, Hadamard-Feynman diagrams on the hyperbolic space H^3.

If we consider the case of a non-zero cosmological constant we have three dimensionfull constants at our disposal,the planck constant $\hbar$, the Newton constant κ and the cosmological scale $L = 1/\sqrt{\Lambda}$. We have two lengths at our disposal: the maximal cosmological length and a minimal planck scale $l_P = \hbar\kappa$ (keep in mind that we have set the speed of light c to 1). We can therefore introduce a dimensionless ratio

$$\frac{k}{\pi} = \frac{L}{\hbar\kappa}. \tag{99}$$

k is then quantized and labels the quantum group deformation parameter $q = \exp\frac{i\pi}{k}$ (see e.g [20]). The two length scales also give rise to two mass scales. The physical implication of these mass and length scales imply that any change of length or mass Δl, Δm is bounded from above and below

$$\hbar\kappa < \quad \Delta l \quad < \pi L, \tag{100}$$

$$\frac{\hbar}{L} < \quad \Delta m \quad < \frac{\pi}{\kappa}. \tag{101}$$

The bounds on Δl imply that l is discrete in Planck unit and bounded from above and similarly the mass is discrete in cosmological units and bounded from above. This is indeed what is realized in

[15]It is often believe that this is true only because we are studying three-dimensional field theory and this effect should disappear in higher dimension because it is impossible to have a non-trivial statistics in $3+1$ dimensions. This relies on the fact that in a classical spacetime the homotopy group of the configuration space of n particles on the sphere is the pure symmetric group. This belief amounts to suppose that such a theorem still holds true in the context of a non-commutative space time. This is far from being obvious and there is no evidence supporting this hypothesis. Moreover anybody who has dealt with non-commutative space-times knows that in a deep non-commutative regime, the notion of dimension is not sharply defined, and our argument tends to be insensitive to dimensionality.

the Turaev-Viro model. It defines a state sum model where the spins j representing the geometrical information label representations of a quantum group $SU(2)_q$ and the summation over j is bounded from above $d_j = 2j+1 < k$. Then the physical length is related to the spin by

$$l = \hbar\kappa d_j. \tag{102}$$

The insertion of spinless particles in this model have been described by Barrett in [7] and are analogous to the insertion of particle in the Ponzano-Regge model. Instead of inserting $\Delta(\theta)\chi_j(h_\theta)$ in the state sum measure along the edge of the Feynman graph, one puts S_{ja} where a is a quantum group representation label (so $d_a < k$). The spin a is related to the physical mass by

$$m = \frac{\hbar d_a}{L} \tag{103}$$

and S is given by

$$S_{ja} = \frac{\sin\frac{\pi}{k}d_j d_a}{\sin\frac{\pi}{k}}. \tag{104}$$

S_{j0} is the quantum dimension. In this model both lengths and masses are discretized and satisfy the bounds (100). The amplitude is

$$\mathcal{Z}_{q,M}(\Gamma_a) = N^{1-|v|}\sum_{\{j_e\}}\prod_{e\notin\Gamma} S_{j_e 0}\prod_{e\in\Gamma} S_{j_e a_e}\prod_t \left\{ \begin{array}{ccc} j_{e_{t_1}} & j_{e_{t_2}} & j_{e_{t_3}} \\ j_{e_{t_4}} & j_{e_{t_5}} & j_{e_{t_6}} \end{array} \right\}_q, \tag{105}$$

where we use the quantum group $6j$ symbol. N is a normalization parameter given by

$$N = \sum_j S_{j0}^2 = \frac{k}{2\sin^2\frac{\pi}{k}}. \tag{106}$$

$|v|$ is the number of vertices of the triangulation. One can show that this definition of the Turaev-Viro amplitude is equivalent to the gauge fixed definition.

Starting from the Turaev-Viro model, we can take several limits of the fundamental constants while keeping physical length and mass fixed:

- *The zero cosmological constant limit* $L\to\infty$:
  In this limit $L\to\infty$ the upper bound on length and the lower bound on mass disappear, we then recover Ponzano-Regge model studied previously where lengths are discrete and unbounded, and masses continuous and bounded.

- *The no gravity limit* $\kappa\to 0$:
  This is the limit we are interested in. In this limit, lengths become continuous and bounded while masses turn out discrete and unbounded.

- *The semi-classical limit* $\hbar\to 0$:
  In this limit both the mass and the lengths are continuous and bounded. It would be interesting to understand better the resulting state sum model.

Note that in all these limits the deformation parameter becomes trivial $q\to 1$. But the simple statement $q\to 1$ is not enough to specify which limit we are considering. We also need to define how the basic quantities like masses and lengths are rescaled in the limit. More precisely, the limits

are defined as the same mathematical limit $q \to 1$, however, physically, we need to specify the units for the physical quantities such as the mass and the length: different choices of units lead to different effective theories.

In the following we work in units $\hbar = 1$ and we consider the no-gravity limit of Turaev-Viro. Let x, y be two points on a 3-sphere of radius L and denote by $l(x, y)$ the distance between them. The Hadamard propagator on a sphere of radius L is given by

$$G_m(x,y) = \frac{\sin ml(x,y)}{\sin l(x,y)}, \quad \left(\Delta + m^2 - \frac{1}{L^2}\right) G_m = 0, \tag{107}$$

Where Δ is the Laplacian on the sphere. If we denote by x, y the corresponding points on the unit sphere SU(2) and by a the half integer such that $d_a = mL$ we see that the Hadamard propagator is just the character $G_m(x, y) = \chi_a(xy^{-1})$. The evaluation of Feynman diagram with insertion of the Hadamard propagator of $\mathcal{S}^3$ is given by

$$\mathcal{Z}(\Gamma, a) = \int \prod_v dx_v \prod_e \chi_{a_e}(x_{t(e)} x_{s(e)}^{-1}) \tag{108}$$

The integrals over x_v can be easily performed, and we are left with the evaluation of the spin network functional at the identity

$$\mathcal{Z}(\Gamma, a) = |\Phi_{(\Gamma_\Delta, a_e)}(1)|^2. \tag{109}$$

We claim that this is the no gravity limit of the Turaev-Viro amplitude. For instance if we consider the tetrahedral graph as an example we have

$$\mathcal{Z}(\Gamma, a) = \int \prod_I dx_I \prod_{I<J} \chi_{a_{IJ}}(x_I x_J^{-1}) = \left\{ \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ a_{34} & a_{24} & a_{23} \end{array} \right\}^2. \tag{110}$$

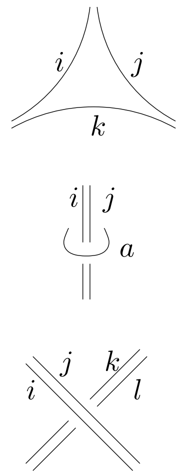
One can check that this is actually the limit $\kappa \to 0$ of the Turaev-Viro amplitude:

$$\mathcal{Z}_q(\Gamma, a) = \frac{1}{N^3} \sum_{\{j_I\}} \prod_{I<J} S_{j_{IJ} a_{IJ}} \left\{ \begin{array}{ccc} j_{34} & j_{24} & j_{23} \\ j_{12} & j_{13} & j_{14} \end{array} \right\}_q^2 = \left\{ \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ a_{34} & a_{24} & a_{23} \end{array} \right\}_q^2. \tag{111}$$

In general, it has been proven by Barrett [8] that $\mathcal{Z}_q(\Gamma, a)$ is given by the square of the quantum group evaluation of the colored graph (Γ, a). The no gravity limit of this evaluation therefore reproduces the Feynman graph evaluation (108).

We are now going to show that the Turaev-Viro amplitude (105) can be written as the Feynman graph evaluation of a non-commutative braided quantum field theory. The classical propagator is a function on $\mathcal{S}^3 \times \mathcal{S}^3$. In a non commutative field theory $\mathcal{S}^3$ becomes a non commutative (fuzzy or q-deformed) space and functions on $\mathcal{S}^3$ becomes operators. If we pick an axis in $\mathcal{S}^3$ and denote R the distance from the north pole of $\mathcal{S}^3$ along this axis, we can describe $\mathcal{S}^3$ as a stack of two spheres of radius $L \sin(R/L)$. Our fuzzy $\mathcal{S}^3$ is then a stack of fuzzy two spheres. It is well known [21] that the space of function on a fuzzy two sphere of a given radius is given by $f_q(\mathcal{S}^2) = \mathrm{End}(V_I)$, where V_I is here the quantum group representation space of spin I. The 'radius' of this q-deformed sphere is given by the quantum dimension $\dim_q(V_I) = \frac{\sin(\pi d_I/k)}{\sin(\pi/k)}$. The space of function on the q-deformed $\mathcal{S}^3$ is then given as a stack of fuzzy spheres

$$F_q(\mathcal{S}^3) \equiv \bigoplus_{I=0}^{(k-2)/2} \mathrm{End}(V_I). \tag{112}$$



$$\equiv \mathrm{Id}_i \otimes \mathrm{Id}_j \otimes \mathrm{Id}_k$$

$$\equiv G_a(i,j)$$

$$\equiv R(i,j,k,l)$$

Figure 3: Feynman rules in the Turaev-Viro model: a Feynman graph evaluation is given by the corresponding Reshetikhin evaluation of the diagram.

A function is thus given by a collection of operators $f(I) \in \mathrm{End}(V_I)$ and the normalized integral is replaced by a q-trace

$$\int_{\mathcal{S}_q^3} f = \frac{1}{N} \sum_I d_I \mathrm{tr}_q(f(I)). \tag{113}$$

The propagator $G_a \in F_q(\mathcal{S}^3) \otimes F_q(\mathcal{S}^3)$ becomes a collection of operators $G_a(I,J) \in \mathrm{End}(V_I \otimes V_J)$ which are given by the Reshetikhin evaluation of the diagram. The Feynman rules are listed in figure 3. For each propagator we insert $G_a(I,J)$, at each vertex we perform the trace and sum over I and for each crossing we insert a R-matrix. This last factor means that, unlike the usual view on non-commutative field theory, the braiding of the graph matters for the Feynman graph evaluation and we are dealing with a braided non-commutative field theory[19]. This expresses the fact that non-commutativity is naturally associated with a change of statistics of the quantum particles.

It is easy to see by performing the sum over I that our Feynman rules gives the evaluation

$$\mathcal{Z}_q(\Gamma, a) = |(\Gamma, a)_q|^2, \tag{114}$$

where $(\Gamma, a)_q$ is the Reshetikhin-Turaev evaluation of the graph Γ colored by a_e. It is less trivial to show that this is equal to the definition (105), this is proven in [8].

6.2 A hyperbolic quantum state sum

We have so far considered Feynman graph evaluation of particle propagating in an Euclidean space having a positive or zero cosmological constant. For completeness we now consider the case where the cosmological constant is negative and the particles propagates in the hyperbolic space H^3.

We claim that the corresponding state sum model is a modification of Turaev-Viro where the deformation parameter is taken to be real $q = \exp(-\frac{\pi}{k})$, no quantization condition holds on k. The representations of $U_q(SU(2))$, $q \in \mathbb{R}$, are labelled by unbounded spins $j = 0, .., \infty$. The state sum model is (up to the normalization factor, the range of summation over j and the gauge fixing)

similar to (105)

$$\mathcal{Z}_{q,M}(\Gamma,\rho) = \sum_{\{j_e\}} \prod_{e\notin T\cup\Gamma} \dim_q(j_e) \prod_{e\in\Gamma} S_{j_e\rho_e} \prod_{e\in T} \delta_{j_e 0} \prod_t \left\{ \begin{array}{ccc} j_{e_{t_1}} & j_{e_{t_2}} & j_{e_{t_3}} \\ j_{e_{t_4}} & j_{e_{t_5}} & j_{e_{t_6}} \end{array} \right\}_q, \tag{115}$$

where T is a maximal tree of $\Delta\backslash\Gamma$, $\dim_q(j) = \frac{\sinh(\pi d_j/k)}{\sinh(\pi/k)}$ is the quantum dimension, ρ is a continuous angle $\in [0,k]$ and

$$S_{j\rho} = \frac{\sin\frac{\pi}{k}\rho d_j}{\sin\frac{\pi}{k}}. \tag{116}$$

We now have the remarkable property that the no-gravity limit of this amplitude is given by the Feynman graph evaluation of Hadamard propagators in H^3. The Hadamard propagator on H^3 is given by

$$G_\rho(x,y) = \frac{\sin\rho l(x,y)}{\rho\sinh l(x,y)}, \tag{117}$$

where $l(x,y)$ is the hyperbolic distance. The Feynman graph evaluation on H^3 is given by

$$Z(\Gamma,a) = \int \prod_v dx_v \prod_e G_{\rho_e}(x_{t(e)}, x_{s(e)}). \tag{118}$$

This amplitude is an evaluation of a colored graph of $\mathrm{SL}(2,\mathbb{C})$. ρ label simple unitary representation of $\mathrm{SL}(2,\mathbb{C})$. The integration at each vertex produces a simple intertwiner of $\mathrm{SL}(2,\mathbb{C})$ according to the general picture described in [22], and the recombination of these intertwiners produces the usual group evaluation of colored graph. This evaluation is exactly the so called relativistic evaluation used as a building block for Lorentzian gravity amplitude [23]. For a tetrahedral graph this amplitude gives the 6j symbol of $\mathrm{SL}(2,\mathbb{C})$

A detailed proof of this claim follows from the results of [24, 25]. One can show that the state sum (115) is an evaluation based on the non-compact quantum group $U_q(\mathrm{SL}(2,\mathbb{C})$. Namely $(\rho,0)$ labels the unitary simple representations of $U_q(\mathrm{SL}(2,\mathbb{C})$. Given such a labelling and a colored graph (Γ,ρ_e), we consider the $U_q(\mathrm{SL}(2,\mathbb{C})$ Reshetikhin-Turaev evaluation $|(\Gamma,\rho_e)|$, the intertwiner between the simple representations at a vertex is constructed in [24]. The no-gravity limit of this evaluation is then given by the evaluation of a graph colored by simple representation of the group $\mathrm{SL}(2,C)$. This proves the claim once we have check that the normalization agree (the interested reader can find more details in [25]).

7 Causality: Hadamard function vs Feynman propagator

7.1 A brief review of Particle Propagators and Feynman Amplitudes

If we would like to recover the (scattering) amplitudes defined by the Feynman diagrams of the field theory, we would like to attach Feynman propagators to links of the graph Γ instead of the Hadamard Green function.

Let us first review the definition of the different Green functions. The basic building blocks are the positive and negative *Wightman functions*. There are solutions to the Klein-Gordon equation defined as:

$$G^{\pm}(x|y) = \frac{1}{(2\pi)^2}\int d^3p\,\theta(p_0)\delta(p^2-m^2)e^{\mp ip\cdot(x-y)}, \tag{119}$$

and expressed in terms of correlations of the scalar field ϕ as:

$$G^+(x|y) = \langle 0|\phi(x)\phi(y)|0\rangle, \qquad G^-(x|y) = \langle 0|\phi(y)\phi(x)|0\rangle = G^+(y|x) = \overline{G^+(x|y)}. \tag{120}$$

The *Hadamard function* is also a solution of the Klein-Gordon equation. Expressed as:

$$G(x|y) = \frac{1}{(2\pi)^2}\int d^3p\,\delta(p^2 - m^2)e^{-ip\cdot(x-y)}, \tag{121}$$

it is the sum $G = G^+ + G^-$. The *Feynman propagator* G_F satisfies the equation $(\Box_x + m^2)G_F = -\delta^{(3)}(x-y)$ and can expressed as:

$$iG_F(x|y) = \theta(x^0 - y^0)G^+(x|y) + \theta(y^0 - x^0)G^-(x|y), \tag{122}$$

$$iG_F(x|y) = \langle 0|T\phi(x)\phi(y)|0\rangle. \tag{123}$$

The proper time expression for G and G_F are:

$$G(x|y) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} dT \int d^3p\, e^{-i[p\cdot(x-y) - T(p^2-m^2)]}, \tag{124}$$

$$\begin{aligned} G_F(x|y) &= -i\frac{1}{(2\pi)^3}\int_0^{+\infty} dT \int d^3p\, e^{-i[p\cdot(x-y) - T(p^2-m^2+i\alpha)]}, \\ &= \frac{1}{(2\pi)^3}\int d^3p\, \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\alpha} \end{aligned} \tag{125}$$

where $\alpha > 0$ is just a regularization.

Finally, one can also introduce the *causal Green function* G_C, which is a solution of the Klein-Gordon equation. Its integral expression is:

$$G_C(x|y) = \frac{-i}{(2\pi)^2}\int d^3p\,\epsilon(p_0)\delta(p^2 - m^2)e^{\mp ip\cdot(x-y)},$$

and correspond to the correlation:

$$iG_C(x|y) = \langle 0|[\phi(x)\phi(y)]|0\rangle = G^+(x|y) - G^-(x|y).$$

Now let us consider a Feynman diagram i.e a graph Γ with Feynman propagators living on its edges. The evaluation of the diagram is the amplitude:

$$I_\Gamma = \int \prod_v d\vec{x}_v \prod_e G^{(F)}_{m_e}(\vec{x}_{t(e)} - \vec{x}_{s(e)}). \tag{126}$$

For a given set of positions $\{\vec{x}_v\}$ and thus a particular (causal) ordering of the vertices, the product of Feynman propagators is simply a product of Wightman functions:

$$\prod_e G^{(F)}_{m_e}(\vec{x}_{t(e)} - \vec{x}_{s(e)}) = \prod_e G^{\epsilon_e}_{m_e}(\vec{x}_{t(e)} - \vec{x}_{s(e)}), \tag{127}$$

where the ϵ_e's record the ordering of the vertices i.e the future/past orientation of each edge. This way, it may seem natural to introduce Wightman diagrams, with a causal structure $\{\epsilon_e\}$ imposed independently from the true ordering of the x's, whose evaluation will be:

$$\begin{aligned} I_\Gamma^{\{\epsilon_e\}} &= \int \prod_v d\vec{x}_v \prod_e G_{m_e}^{\epsilon_e}(\vec{x}_{t(e)} - \vec{x}_{s(e)}) \\ &= \int \prod_e d^3\vec{p}_e \int \prod_v d^3\vec{x}_v\, \theta(\epsilon_e p_e^0)\delta(p_e^2 - m_e^2) e^{-i\vec{p}_e\cdot(\vec{x}_{t(e)} - \vec{x}_{s(e)})} \\ &= \int \prod_e d^3\vec{p}_e\, \theta(\epsilon_e p_e^0)\delta(p_e^2 - m_e^2) \prod_v \delta\left(\sum_{e|v=t(e)} \vec{p}_e - \sum_{e|v=s(e)} \vec{p}_e \right). \end{aligned} \tag{128}$$

Because of the constraint imposing momentum conservation at the vertices, one can solve these constraints as explained previously and pass to the dual formulation of these amplitudes: they can be easily reproduced by a spin foam model.

In order to do the same with the Feynman amplitude, one should get rid of the explicit use of the causal ordering. The solution is to use the proper time formula (125) for G^F which leads to:

$$I_\Gamma = \int \prod_e d^3\vec{p}_e \int_0^{+\infty} dT_e\, e^{iT_e(p_e^2 - m_e^2 + i\alpha)} \prod_v \delta\left(\sum_{e|v=t(e)} \vec{p}_e - \sum_{e|v=s(e)} \vec{p}_e \right). \tag{129}$$

Now, as this includes the momenta conservation constraints, this amplitude can be re-casted in terms of the dual (face) variables and thus easily written in terms of spin foam amplitudes.

In the Euclidean context considered up to now, we ave been working with the Hadamard propagators. In order to properly recover quantum field theory amplitudes, one needs to construct (particle) observables which would reproduce the Wightman propagators and the Feynman propagators. These notions being closely related to causality are more likely to be properly implemented in a Lorentzian model only. The Lorentzian analog of the Ponzano-Regge model has been considered in [26] and is known to be based on the group $\mathrm{SU}(1,1)$. Therefore, we will study in the following how to write a (well-defined) Lorentzian spin foam model based on $\mathrm{SU}(1,1)$, its abelian limit as a spin foam model over the abelian group M_3 (Minkowski space) and causality oriented particle insertions. Discussion of implementation of causality in spin foam models can be found in [27].

An interesting feature of the Lorentz group $\mathrm{SU}(1,1)$ is that there already exists a split between some so-called future oriented timelike representations and past oriented timelike representations. Moreover, the corresponding characters is then $\exp(\mp i d_j \theta)$ instead of $\sin(d_j\theta)$: their abelian limit will directly be the Wightman functions instead of the Hadamard functions. As we will see below, there is also a very natural implementation of the Feynman propagator.

7.2 The abelian limit of the $\mathrm{SU}(1,1)$ characters

$\mathrm{SU}(1,1)$ has two Cartan subgroups. The first one corresponds the (space) rotation and its elements are of the form ($0 \le \theta \le 2\pi$):

$$h_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

while the second one is made of the boosts ($t \in \mathbb{R}$)

$$\pm a_t = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Then the (unitary) representations of $\mathrm{SU}(1,1)$ are also of two kinds. We first have the continuous series of representations, labelled by $s \in \mathbb{R}$. They correspond to the (co-adjoint) orbits defined by the one-sheet hyperboloid in Kirillov's theory, The characters are:

$$\chi_s(h_\theta) = 0 \qquad \chi_s(\pm a_t) = \frac{\cos st}{|\sinh t|}.$$

Then we have two discrete series of representations -positive and negative series- whose corresponding orbits are the upper and lower hyperboloid. We label them by an integer $n \geq 1$. The characters are:

$$\chi_n^\pm(h_\theta) = \frac{\mp e^{\pm i(n-1)\theta}}{2i\sin\theta} \qquad \chi_n^\pm(a_t) = \frac{e^{-(n-1)|t|}}{2|\sinh t|} \qquad \chi_n^\pm(-a_t) = (-1)^n \frac{e^{-(n-1)|t|}}{2|\sinh t|}.$$

Now let us consider the abelian/classical/flat limit of the characters for the (positive) discrete representations. We take the angles $\theta = m\kappa$ (or $t = m\kappa$) going to zero while we take the representation labels $(n-1) = l/\kappa$ to ∞ by taking the limit $\kappa \to 0$. In this abelian limit, group elements of $\mathrm{SU}(1,1)$ are mapped into Lie algebra vectors of $\mathfrak{su}(1,1) \sim M_{(3)}$: rotations correspond to time-like vectors and boosts to space-like vectors. The limit of the character reads:

$$\chi_n^+(h_\theta) \to \tilde{\chi}_l(x) = \frac{-e^{il|x|}}{2i|x|} \text{ for } \vec{x} \text{ timelike,} \tag{130}$$

and

$$\chi_n^+(a_t) \to \tilde{\chi}_l(x) = \frac{e^{-l|x|}}{2|x|} \text{ for } \vec{x} \text{ spacelike.} \tag{131}$$

Then from the point of view of the abelian limit of the Lorentzian model, it is natural to consider Feynman evaluation with propagators given by $G_m(\vec{x}) = \tilde{\chi}_m(x)$. Using the Kirillov formula:

$$\forall \vec{x} \in M_3,\, G_m(\vec{x}) = m \int_{\mathcal{H}_+} d^2\vec{n}\, e^{im\vec{x}\cdot\vec{n}} = \frac{1}{m} \int_{\mathcal{C}_+} d^3\vec{p}\, \delta(|p| - m) e^{i\vec{x}\cdot\vec{p}}, \tag{132}$$

where $\mathcal{H}_+$ is the future light cone and $\mathcal{C}_+$ the future cone. Comparing this expression with the definition (119), it is obvious that the propagator defined as the abelian limit of the character χ_n^+ is the positive Wightman (Green) function. Similarly, if we would assign a negative representation n^- to an edge of the graph, we will put assign the negative Wightman function to that link. Finally, we could assign the representation $n^+ \oplus n^-$ to the edge, the corresponding character would be the sum $\chi_n^+ + \chi_n^-$ which would lead to the usual Hadamard (Green) function.

7.3 Constructing a Lorentzian spin foam model

Let us start by constructing an abelian Lorentzian spin foam model, with observables for particle insertion which exactly reproduce the Feynman amplitudes. Then we would like to write a Lorentzian version of the Ponzano-Regge model, based on $\mathrm{SU}(1,1)$ whose abelian limit (corresponding to the Newton constant going to 0) reduces to that model.

Formally, we can define the abelian Lorentzian spin foam exactly the same way than the Euclidean one:

$$Z = \prod_{\bar{e}} \int_{M_3} d^3\vec{u}_{\bar{e}} \prod_{\bar{f}} \delta(\vec{u}_{\bar{f}}), \tag{133}$$

where we now label the (dual) edges by vectors in the 3-dimensional Minkowski space M_3. To insert a particle on an edge e, we identify the momentum $\vec{p}_e$ with the geometric variable $\vec{u}_{\bar{f}}$ living on the dual plaquette. Then:

- to insert the Hadamard function: one replaces the constraint $\delta(\vec{u}_e)$ by
$$\delta(|\vec{u}_e|^2 - m^2).$$
- to insert a Wightman function, one splits the $\delta(|\vec{u}_e|^2 - m^2)$ in two depending on the time orientation and uses
$$\theta(\epsilon u_e^0)\, \delta(|\vec{u}_{\bar{f}}|^2 - m^2).$$
- to insert the Feynman propagator, one splits $\delta(|\vec{u}_e|^2 - m_e^2)$ in two in the proper time formulation and uses
$$\int_0^{+\infty} dT_e e^{iT_e(|\vec{u}_e|^2 - m_e^2 + i\alpha)},$$
where α is just a regularization.

Now we would like to introduce the corresponding propagators in the non-abelian case, i.e. the observables describing particles in the Lorentzian spin foam model. Technically, our aim is to identify the quantity "$\Delta(\theta)\delta_\theta(g)$" which we should substitute to $\delta(g)$ in the spin foam model. Our main criteria is to check that the non-abelian propagators have the right expected behavior in the no-gravity limit $\kappa \to 0$.

Let us start by constructing the Hadamard propagator. The non-abelian counterpart of $\delta(|\vec{u}_e|^2 - m^2)$ is to fix the group element g_e in the same conjugacy class than the Cartan element h_θ. Thus we introduce the distribution $\delta_\theta(g)$ defined by:

$$\int_{\mathrm{SU}(1,1)} dg\, \delta_\theta(g) f(g) = \int_{\mathrm{SU}(1,1)/\mathrm{U}(1)} du\, f(u h_\theta u^{-1}),\ \forall f. \tag{134}$$

Let us insist at this point that h_θ, $\theta \in]0,\pi]$ is not conjugated to $h_{-\theta}$. Then as $\mathrm{SU}(1,1)/\mathrm{U}(1) \sim \mathcal{H}_+ \cup \mathcal{H}_-$ is the (disjoint) union of the upper and lower mass-shell hyperboloid, we can actually write:

$$\int_{\mathrm{SU}(1,1)} dg\, \delta_{\theta>0}(g) f(g) = \int_{\mathcal{H}_+} du\, f(u h_\theta u^{-1}), \quad \int_{\mathrm{SU}(1,1)} dg\, \delta_{-\theta}(g) f(g) = \int_{\mathcal{H}_-} du\, f(u h_\theta u^{-1}).$$

It is easy to check that

$$\delta_\theta(g) = \sum_{n,\epsilon} \chi_n^{-\epsilon}(h_\theta) \chi_n^\epsilon(g), \tag{135}$$

since the character of the continuous representations vanish on the U(1) Cartan elements h_θ. The Hadamard propagator is then defined as:

$$\delta_\theta^{Hadamard}(g) \;\equiv\; \delta_\theta(g) + \delta_{-\theta}(g). \tag{136}$$

In the abelian limit, $\theta \in [0,\pi]$ will be put it κ units, $\theta = m\kappa$, with m the (renormalized) mass. Notice that $\chi_n^{\pm}(-\theta) = \chi_n^{\mp}(\theta)$, so that we can write:

$$\delta_\theta^{(H)}(g) = \sum_{n,\epsilon,\epsilon'} \chi_n^\epsilon(\theta)\chi_n^{\epsilon'}(g). \tag{137}$$

Then the Wightman propagators only impose $p^0 > 0$ or $p^0 < 0$. This is exactly whether $u \in \mathcal{H}_+$ or $u \in \mathcal{H}_-$, i.e. distinguishing $\theta > 0$ from $\theta < 0$. Thus we define the non-abelian Wightman propagators for $\theta \in]0,\pi]$:

$$\delta_\theta^{(W+)}(g) \equiv \delta_\theta(g), \qquad \delta_\theta^{(W-)}(g) \equiv \delta_{-\theta}(g). \tag{138}$$

The hard part is to define the non-abelian equivalent of the Feynman propagator. Looking at the proper time expression (125) of the propagators, we can interpret (137) as the equivalent expression of the Hadamard Green function in the non-abelian case. Then the Feynman propagator should be a particular splitting of the sum over ϵ's. To decide, let us look at the abelian limit of $\sum_n \chi_n^\epsilon(\theta)\chi_n^{\epsilon'}(g)$. Using the explicit formula for the characters, it is easy to get ($|\vec{p}| \in \mathbb{R}_+$):

$$\sum_n \Delta(\theta)\chi_n^\epsilon(\theta)\chi_n^{\epsilon'}(g) \underset{\kappa\to 0}{\sim} \int_0^{+\infty} dl\, \frac{-\epsilon\epsilon'}{4}\frac{sg(p_0)}{|\vec{p}|} e^{il(\epsilon' sg(p_0)|\vec{p}|+\epsilon m)}, \tag{139}$$

when $g = \cos\phi + isg(p_0)\sin\phi\hat{u}.\vec{\sigma}$, $\phi \in [0,\pi]$, $\hat{u}$ normalized positive timelike vector, is conjugated to the compact Cartan group. Here we have defined $\kappa\vec{p} = sg(p_0)\sin\phi\hat{u}$ for $\kappa \to 0$. If g is conjugated to the non-compact Cartan group, it corresponds to a spacelike momentum and the limit is (still $|\vec{p}| \in \mathbb{R}_+$):

$$\sum_n \Delta(\theta)\chi_n^\epsilon(\theta)\chi_n^{\epsilon'}(g) \underset{\kappa\to 0}{\sim} \int_0^{+\infty} dl\, \frac{-\epsilon}{4i}\frac{1}{|\vec{p}|} e^{-l(|\vec{p}|-i\epsilon m)}.$$

Inserting a regulator $\alpha > 0$ in the expression (139), we get:

$$\begin{aligned}\sum_n e^{-(n-1)\alpha}\Delta(\theta)\chi_n^\epsilon(\theta)\chi_n^{\epsilon'}(g) &\underset{\kappa\to 0}{\sim} \int_0^{+\infty} dl\, \frac{-\epsilon\epsilon'}{4}\frac{s}{|\vec{p}|} e^{il(\epsilon' sg(p_0)|\vec{p}|+\epsilon m+i\alpha)}, \\ &\underset{\kappa\to 0}{\sim} -\frac{\epsilon\epsilon'}{4}\frac{s}{|\vec{p}|}\frac{i}{\epsilon' s|\vec{p}|+\epsilon m+i\alpha},\end{aligned} \tag{140}$$

where we note $s = sg(p_0)$.

A priori, one would say that combining these expression, one can obtain all the different propagators of quantum field theory with different choices and signs of the poles. It is actually true except for the case of the Feynman propagator whose case is slightly more subtle.

Indeed, we can combine

$$\frac{1}{s|p|-m+i\alpha} + \frac{1}{s|p|+m+i\alpha} = 2s|p|\,\frac{1}{|p|^2-m^2+is\alpha},$$

which would give the Feynman propagator if $s\alpha > 0$. Therefore the sign of the regulator would need to depend on the sign s of the group element g. The easiest way out is to introduce a group element $|g|$ defined by:

$$g = uh_\phi u^{-1} = \cos\phi Id + i\sin\phi\hat{u}.\vec{\sigma} \Rightarrow |g| = uh_{|\phi|}u^{-1} = \cos\phi Id + i|\sin\phi|\hat{u}.\vec{\sigma},$$

so that always $s(|g|) = +1$. Then it is straightforward to check that a good choice of Feynman propagator is:

$$\delta_\theta^{(F)}(g) \equiv 2i \sum_n e^{-(n-1)\alpha} \Delta(\theta) \chi_n^-(\theta) \chi_n(|g|) \equiv 2i \sum_n \chi_n(|g|)(\Delta(\theta - i\alpha)\chi_n^-(\theta - i\alpha)). \tag{141}$$

since its classical limit is the expected $1/(|p|^2 - m^2 + i\alpha)$.

Conclusion

In this paper we have shown how the quantum gravity amplitudes for particles coupled to three dimensional gravity proposed in [3] are related to the usual amplitude in the no-gravity limit. More surprisingly, it turns out that these amplitudes can be understood as coming from a non-commutative braided quantum field theory.

We have seen however that the natural quantum gravity amplitudes amount to insert Hadamard propagators, then we have discussed how the insertion of a Feynman propagator can be naturally implemented in the spin foam model at least in the Lorentzian case. What would need to be done in this context is to show that this procedure is really well-defined in the spin foam context, namely that our proposal for introducing the Feynman propagator in the Lorentzian Ponzano-Regge model respect the topological invariance of the theory and leads to triangulation independent amplitudes. Also it would be interesting to understand more generally the definition of the Feynman propagator in the case of non-zero cosmological constant involving quantum group.

It is not totally surprising that the effective field theory describing the dynamics of particles coupled to three space-time dimensions is given in term of a non-commutative braided quantum field theory. Indeed it is known that particles in 3d gravity respect the principle of Doubly special relativity and that the corresponding algebra of symmetry is a κ-deformation of the Poincaré group. Quantum field theories based on κ-deformed Poincaré group have been already considered in the literature [28]. Usually there is an ambiguïty in the definition of the spacetime coordinates which leads to different proposals, none of which are physically motivated. Our analysis gives a unique prescription and solves this ambiguïty. It seems that the explicit form of the action and star product which we found has never been proposed. An interesting perspective appear if one remark that the structure of the effective field theory which arise here possess some strong similarity with the Group field theories that appears in the spin foam quantization of gravity [30], it is tempting to conjecture that this similarity is not purely accidental and that the structure which is unravelled here should generally appear in the spin foam context.

This questions becomes more interesting if we think of 2+1 gravity as a toy model for 3+1 gravity and hope to learn something about the effective dynamics of matter coupled to quantum gravity. This hope is substantiated by the recent work [29] which shows a reformulation of 3+1 gravity in terms of a perturbed BF theory where matter coupling to gravity is very similar to the 2+1 gravity case. Moreover, it seems that the star product that we have introduced should be generalized to a 3+1 quantum space-time. A subtle point is that star products in the field theory action would then also explicitly involve polynomial terms in the coordinates (or equivalently derivations in the momenta), which are usually ignored in standard analysis of possible non-commutative field theories as effective quantum gravity theories. This specific structure allows the field theory to be actually invariant under the deformation of the Poincaré group.

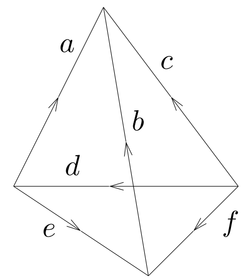


Figure 4: Tetrahedral net as the skeleton of a 2-sphere.

Acknowledgements

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A Examples of the duality transform of Feynman diagrams

A.1 Example of $\Gamma \hookrightarrow \mathcal{S}^2$

Let us consider the tetranet embedded in $\mathcal{S}^2$. We have six edges thus six (momentum) variables $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$, and four vertices resulting in four constraints:

$$\begin{cases} a+b+c=0 \\ a-d+e=0 \\ c+d+f=0 \\ -b+e+f=0 \end{cases} \tag{142}$$

The rank of this system is 3, so that we need at least $6-3$ variables to parametrize its solutions. A convenient parametrization is given by assigning a new variable to each face, thus four variables $\alpha, \beta, \gamma, \delta$:

$$a = \alpha - \delta, \quad b = \beta - \alpha, \quad c = \delta - \beta, \quad d = \gamma - \delta, \quad e = \gamma - \alpha, \quad f = \beta - \gamma. \tag{143}$$

It is obvious that we can translate all face variables by a constant vector $\vec{k}$ and it wouldn't change anything. So, at the end of the day, we truly have 3 parameters to describe the space of solutions, which is consistent.

More formally, one can introduce conjugate variables to the face variables $\vec{\alpha}, \vec{\beta}, \ldots$. Let us note them $\vec{v}_\alpha, \ldots$. Then $\vec{C} = \vec{v}_\alpha + \vec{v}_\beta + \vec{v}_\gamma + \vec{v}_\delta$ generates simultaneous translations on $\vec{\alpha}, \ldots$, so that we can see the residual invariance as imposed by the constraint $\vec{C} = 0$. This can be understood as the closure constraints for the tetrahedron assuming that the vectors $\vec{v}_\alpha, \ldots$ are the normal vectors to the faces of the tetrahedron.

A.2 Example of $\Gamma \hookrightarrow T^2$

One can also embed the tetranet in the torus T^2. As the graph is the same as in the previous case, we have the same variables with the same constraints. The difference is that we now have only two

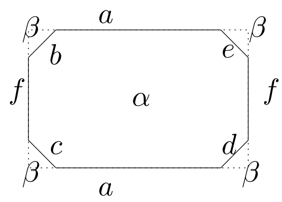

Figure 5: Tetrahedral net as the skeleton of a 2-torus.

faces $\vec{\alpha}, \vec{\beta}$. This is not enough variables to parametrize the space of solutions. We need two extra vectors, which will be associated to the two (non-contractible) cycles of the torus:

$$\left\{\begin{array}{ccc} a & = & \mathcal{C}_1 \\ b & = & (\alpha-\beta)-\mathcal{C}_1 \\ c & = & (\beta-\alpha) \\ d & = & (\alpha-\beta)-\mathcal{C}_2 \\ e & = & (\alpha-\beta)-\mathcal{C}_1-\mathcal{C}_2 \\ f & = & \mathcal{C}_2 \end{array}\right.$$

The chosen cycles, as sequences of edges, are $a, -b, -e$ and $-d, -e, f$. Let us also notice that once again we can translate $\vec{\alpha}, \vec{\beta}$ simultaneously by any vector without changing anything. This corresponds to the closure of the faces.

A.3 Example of $\Gamma \hookrightarrow \mathcal{S}^2 \hookrightarrow \mathcal{S}^3$

Let us consider the 3d manifold $\mathcal{S}^3$ and triangulate it using two tetrahedra. The skeleton of the resulting triangulation Δ is simply a tetranet, which we take as on fig.4. Let us then consider the Feynman diagram Γ made of the edges a, b, c, d, e. The graph has four vertices and the corresponding equations are:

$$b = e, \quad c + d = 0, \quad a + b + c = 0, \quad a + e = d. \tag{144}$$

To parametrize the space of solutions, we introduce embed Γ in the tetranet considered as a triangulation of $\mathcal{S}^2$. The tetranet has four faces for each of which we introduce a new variable: u_1 for abe, u_2 for acd, u_3 for bcf and u_4 for def. Then we can express the original momentum variables $a, .., e$ in terms of these face variables:

$$a = u_1 - u_2, \quad b = u_3 - u_1, \quad c = u_2 - u_3, \quad d = u_4 - u_2, \quad e = u_4 - u_1.$$

For this to be a solution of the constraints (144), it is necessary and sufficient to impose $f = u_3 - u_4 = 0$. And we finally recover the formula (10) for the Feynman amplitude.

B Feynman graph with Spinning particles

B.1 Overview of the spinning particle

In [3] Ponzano-Regge amplitudes for spinning and spinless particles have been proposed. In the main text we have shown that the spinless Ponzano-Regge amplitudes reproduces in the no gravity

limit the usual Hadamard-Feynman amplitudes. In this appendix we show that the same is true for the spinning amplitudes. Let us present a quick review of the properties and propagator for a massive spinning particle/field in three dimensions. Let us start by the Euclidean case, keeping that all the considerations could be straightforwardly translated to the Minkowskian case.

Let us look at a one-particle state $\psi_{p,\sigma}$ where p denotes the (3-)momentum and σ the other degrees of freedom:

$$P^\mu \psi_{p,\sigma} = p^\mu \psi_{p,\sigma}.$$

Considering the action of Lorentz transformations $U(\Lambda)$ on the translation generators, $U(\Lambda)$ maps a state of momentum p onto a state of momentum Λp:

$$U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma',\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'}.$$

Let now us choose a reference momentum k^μ and define states $\psi_{p,\sigma}$ from the states $\psi_{k,\sigma}$:

$$\psi_{p,\sigma} = N(p)U(s(p))\psi_{k,\sigma},$$

where $N(p)$ is a normalization factor and $s(p)$ maps k to p. For a massive particle, k is usually taken as $p^{(0)} = (1,0,0)$. The little group of Lorentz transformations W leaving k invariant is the rotation group U(1). So $s(p)$ is a section[16] from SU(2)/U(1) to SU(2) such that:

$$s(p).(1,0,0) = \frac{1}{m}p = \left(\sqrt{1-\frac{|\vec{p}|^2}{m^2}}, \frac{1}{m}\vec{p}\right).$$

Then the action of a Lorentz transformation reads:

$$U(\Lambda)\psi_{p,\sigma} = N(p)U(s(\Lambda p))U(W(\Lambda,p))\psi_{k,\sigma},$$

with $W(\Lambda,p) = s(\Lambda p)^{-1}\Lambda s(p)$ living in the little group U(1). Note that $W(R,p^{(0)}) = R$ for rotations and $W(B,p^{(0)}) = Id$ for pure boosts. From this, it is straightforward to check that postulating the action of the little group:

$$U(W)\psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma',\sigma}(W)\psi_{k,\sigma'},$$

we get the action of an arbitrary Lorentz transformations:

$$U(\Lambda)\psi_{p,\sigma} = \left(\frac{N(p)}{N(\Lambda p)}\right)\sum_{\sigma'} D_{\sigma',\sigma}(W(\Lambda,p))\psi_{\Lambda p,\sigma'}.$$

So that instead of giving all the coefficients $C_{\sigma',\sigma}(\Lambda)$, it is enough to define the coefficient $D_{\sigma',\sigma}(W)$: from a representation of the little group U(1), we induce a representation of the Lorentz group SU(2). The usual normalization is $N(p) = \sqrt{k_0/p_0}$ so that the scalar product is normalized $\langle\psi_{p,\sigma},\psi_{p,\sigma}\rangle = \delta_{\sigma\sigma'}\delta^{(2)}(\vec{p}-\vec{p'})$. The *spin* s is the choice of a irreducible representation of the little group U(1), which defines the coefficient $D_{\sigma',\sigma}$, and σ is the angular momentum.

[16] We choose

$$s(m\cosh\theta, m\sinh\theta\cos\phi, m\sinh\theta\sin\phi) = e^{i\phi\sigma_3}e^{\theta\sigma_2}. \qquad (145)$$

All this can be directly translated to a field theory. Let us assume that the field ψ has components ψ_l labelled by l and is a representation of the Poincaré group. More precisely, we split the field into creation and annihilation parts $\psi_l = \psi_l^+ + \psi_l^-$:

$$\psi_l^+ = \sum_\sigma \int d^2\vec{p}\, u_l(x,\vec{p},\sigma)a(\vec{p},\sigma), \qquad \psi_l^- = \sum_\sigma \int d^2\vec{p}\, v_l(x,\vec{p},\sigma)a^\dagger(\vec{p},\sigma).$$

Then the action of a Poincaré transformation reads:

$$U(\Lambda,a)\psi_l^\pm U(\Lambda,a)^{-1} = \sum_{l'} D^I_{ll'}(\Lambda^{-1})\psi_{l'}^\pm(\Lambda x + a),$$

where we have chosen a representation I os the Lorentz group SU(2). Interactions for such a field can be written as:

$$V(x) = g_{l_1\dots l_N l'_1 \dots l'_M}\psi^-_{l'_1}(x)\dots\psi^-_{l'_M}(x)\psi^+_{l_1}(x)\dots\psi^+_{l_N}(x),$$

where $g_{l_1\dots}$ is an intertwiner for the Lorentz group (in our case where the Lorentz group is SU(2), it is a Clebsh-Gordan coefficient), so that the interaction is a scalar for the Lorentz group. Then the coefficients u_l (and v_l) defining the field satisfy the following constraints:

$$u_l(x,\vec{p},\sigma) = (2\pi)^{-3/2}e^{ip\cdot x}u_l(\vec{p},\sigma),$$

$$\sum_{\sigma'} u_{l'}((\vec{\Lambda p}),\sigma')D^s_{\sigma'\sigma}(W(\Lambda,p)) = \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_l D_{l'l}(\Lambda)u_l(\vec{p},\sigma).$$

In order to understand the structure of this constraint, let us split it in the two cases when Λ is a pure boost or a rotation. Λ boost, with $W(\Lambda,p)=1$, allows to map the reference momentum ${}^{(2)}\vec{p}=0$ to an arbitrary momentum:

$$u_{l'}(p,\sigma) = \sqrt{\frac{m}{p^0}}\sum_l D_{l'l}(s(p))u_l(0,\sigma),$$

so that we go from $u_l(0,\sigma)$ to $u_l(p,\sigma)$ simply by a boost. Then for a rotation $\Lambda = R$, with $W(R,p)=R$ and ${}^{(2)}\vec{p} = (\vec{Rp}) = 0$, we have:

$$\sum_{\sigma'} u_{l'}(0,\sigma')D^s_{\sigma'\sigma}(R) = \sum_l D^I_{l'l}(R)u_l(0,\sigma),$$

which states that the coefficient $u_l(0,\sigma)$ can be understood as an intertwiner between the representation of U(1) defined by the spin s and the representation of the little group U(1) induced by the representation I of the Lorentz group SU(2). So that the field definitively lives in the representation of spin s of the little group. In our Euclidean 3d context, I is a representation of SU(2) and s, the spin, a representation of $U(1)$ such that $s \le I$ (i.e $s \hookrightarrow I$). There is only one value of σ which is s and $D^s(R(\theta)) = \exp(is\theta)$ for a rotation of angle θ. Then the unique solution to the constraints is $u_l(0) = \langle l|s\rangle = \delta_{ls}$.

Finally, the causal propagator is defined as the commutator of the field ψ with itself. Assuming that $v_l = u_l^\dagger$, we have:

$$[\psi(x),\psi(y)] = \frac{1}{(2\pi)^3}\int d^2\vec{p}\,\Delta_{ll'}(p)e^{ip\cdot(x-y)}, \qquad \text{with} \quad \Delta_{ll'}(p) = \sum_\sigma u_l(\vec{p},\sigma)u^\dagger_{l'}(\vec{p},\sigma).$$

To sum up, the propagator $\Delta^I_{ll'}(p)$ of a field of spin s must satisfy the following properties:

- $\Delta(\Lambda p \Lambda^{-1}) = \Lambda \Delta(p) \Lambda^{-1}$.
- $\Delta(p^{(0)})$ is the projector $|s\rangle\langle s|$ to the right spin.
- $\Delta(p)$ is a projector: $\Delta(p) = \Delta(p)\Delta(p)$.

I represents the total angular momentum while s, the spin, is only the intrinsic rotation of the particle/field: $|s\rangle$ is a vector of the I representation i.e $s \leq I$. From the field point of view, $I = s$ represents the fundamental field $\phi^{(s)}$ of spin s, while the representation $I = s + k$ corresponds to the k-th derivative of the field $\partial^k \phi^{(s)}$.

The unique solution to these constraints is:

$$\Delta(p) = D^I\left(s(p)\right)|s\rangle\langle s| D^I\left(s(p)\right)^{-1}. \tag{146}$$

This is easy to generalize to different I's on the left and right hand side. Now

$$\Delta(p) = D^I\left(s(p)\right)|s\rangle\langle s| D^{I'}\left(s(p)\right)^{-1}$$

corresponds to the correlation $\langle \partial^k \phi^{(s)}\, \partial^{k'} \phi^{(s)} \rangle$ with $I = s + k, I' = s + k'$.

For $I = s = 0$, the scalar field, the propagator is obviously $\Delta = 1$. For the spin-1/2 particle, $I = s = 1/2$, the definition of the section $s(p)$ reads in spinor notations[17]:

$$D^{\frac{1}{2}}(s(p)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D^{\frac{1}{2}}(s(p))^{-1} = \frac{{}^{(3)}\vec{p}}{m} \cdot \vec{\sigma} = \frac{1}{m} \begin{pmatrix} p^0 & p_+ \\ p_- & -p^0 \end{pmatrix},$$

where the σ_i's are the Pauli matrices normalized to $\sigma_i \sigma_i = Id$. So the propagator for $s = +1/2$ is:

$$\Delta_{+1/2}(p) = D^{\frac{1}{2}}(s(p)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D^{\frac{1}{2}}(s(p))^{-1} \tag{147}$$

$$= D^{\frac{1}{2}}(s(p)) \frac{1}{2} \left(Id + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) D^{\frac{1}{2}}(s(p))^{-1} = \frac{1}{2m}\left(m + \vec{p} \cdot \vec{\sigma}\right), \tag{148}$$

and similarly for the negative helicity $s = -1/2$:

$$\Delta_{+1/2}(p) = \frac{1}{2m}\left(m - \vec{p} \cdot \vec{\sigma}\right). \tag{149}$$

And we recognize the usual expressions for a spinor field for the two helicities.

For a $I = 1$ field, the section is defined by:

$$D^{I=1}(s(p)). \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} p_0 \\ p_x \\ p_y \end{pmatrix} \quad \Rightarrow \quad D(s(p)) = R(\widehat{p}) B(|^2\vec{p}|) R(\widehat{p})^{-1},$$

[17] The spinor representation of a 3-vector (u_0, u_x, u_y) is:

$$\begin{pmatrix} u_0 & u_+ = u_x + i u_y \\ u_- = u_x - i u_y & -u_0 \end{pmatrix}.$$

with

$$R(\widehat{p}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{p_x}{|^2\vec{p}|} & -\frac{p_y}{|\vec{p}|} \\ 0 & \frac{p_y}{|\vec{p}|} & \frac{p_x}{|\vec{p}|} \end{pmatrix}, \qquad B(|^2\vec{p}|) = \begin{pmatrix} \frac{p_0}{m} & \frac{|\vec{p}|}{m} & 0 \\ -\frac{|\vec{p}|}{m} & \frac{p_0}{m} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we look at the spin $s = 0$ field, the projector is:

$$|I = 1, 0\rangle\langle 0| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that the propagator is

$$\Delta^{I=1,s=0}_{\mu\nu}(p) = \frac{p_\mu p_\nu}{m^2},$$

which naturally corresponds to the correlation $\langle \partial_\mu \phi \partial_\nu \phi \rangle$ of a scalar field ϕ. To recover the usual propagator for a spin-1 field, $s = 1$, we in fact sum over the states $|s = -1\rangle$ and $|s = +1\rangle$. The projector then reads:

$$|-\rangle\langle -| + |+\rangle\langle +| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that the propagator is simply:

$$\Delta^{I=s=1}_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}. \tag{150}$$

B.2 Inserting spinning particles in the spin foam

Now, let us look at the Feynman amplitude of a graph with massive and spinning particles. The part concerning the mass will not change: we still impose the constraint $\delta(|\vec{p}| - m)$ on each edge and conservation of the momentum at each vertex. Moreover we now need to take into account the further degrees of freedom corresponding to the spin (or intrinsic rotation) of the particles. On each edge e, we have a spin s_e and we put the corresponding propagator $\Delta^{I=s_e}(p_e)$. At the vertices, we need to take into account the interaction between spinning particles: we include $SU(2)$ intertwiners intertwining between the representations $I = s_e$ of the edges meeting at the vertices. On the whole, the Feynman evaluation reads:

$$I_\Gamma = \int \prod_e d^3\vec{p_e} \prod_e \frac{\delta(|p_e| - m_e)}{4\pi m_e} \Delta^{I_e=s_e}_{l_e l'_e}(p_e) \times \prod_v \delta\left(\sum_{e|v=t(e)} \vec{p}_e - \sum_{e|v=s(e)} \vec{p}_e \right) C^{I\dots}_{l\dots}. \tag{151}$$

Let us point out that we need not in principle take $I_e = s_e$, then the propagators would live in the representations I_e and the intertwiners C intertwine between the I_e's: the spins s_e would only enter as projectors in the definition of the propagators $\Delta_{ll'}(p) = D_{ls}(s(p))D_{sl'}(s(p))^{-1}$.

It is straightforward to write this amplitude in its dual form solving the constraints imposed by momentum conservation at the vertices. Indeed, adding spins doesn't modify anything, so that the amplitude corresponding to the graph embedded in the 3 triangulation reads:

$$I_\Gamma = \int \prod_{\bar{e}} d^3\vec{u}_{\bar{e}} \prod_{e \notin \Gamma} \delta(u_e) \prod_{e \in \Gamma} \frac{\delta(|u_e| - m_e)}{4\pi m_e} \Delta^{I_e=s_e}_{l_e l'_e}(p_e) \times C^{I\dots}_{l\dots}. \tag{152}$$

So given the graph Γ, we identify the plaquettes dual to the edge of Γ. These form a tube around the graph. We evaluate the holonomy p_e around each plaquette, then joining them all into a spin network -a graph drawn on the surface of the tube- using the intertwiners $C^{I\cdots}_{l\cdots}$. Finally the Feynman evaluation id the product of the mass shell conditions with the spin network evaluation.

Let us now transpose this formula to the non-abelian case in order to encode spinning particles in the Ponzano-Regge model, such that its abelian limit will reduce to the above Feynman amplitude.

The mass shell condition translates into $\delta_{\theta_e}(g_e)$, which imposes that $g_e = uh_\theta u^{-1}$ for some $u \in \mathrm{SU}(2)/\mathrm{U}(1)$. u represents the momentum of the particle so that the spinning propagator translates to $D^{I=s}_{ls}(u)D^{I=s}_{sl'}(u^{-1})$. Finally, we contract these propagators using the intertwiners C's which defines the physical content of the quantum field theory we would like to encode on the spin foam. The final amplitude is:

$$Z_{\Gamma,\{\theta_e,s_e\}} = \int \prod_{\bar{e}} dg_{\bar{e}} \prod_{e\notin\Gamma} \delta(g_e) \prod_{e\in\Gamma} \frac{V_H}{V_G\Delta(\theta_e)} \int_{\mathrm{SU}(2)/\mathrm{U}(1)} du_e \delta(guh_\theta u^{-1}) D^{I=s_e}_{l_e s_e}(u) D^{I=s_e}_{s_e l'_e}(u^{-1}) \times C^{I\cdots}_{l\cdots}. \tag{153}$$

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